

CHAPTER

07

Gravitation

Planetary Motion and the Law of Gravitation • Gravity, Field and Potential
• Escape Speed and Satellites

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READING KEY

DEFINITION

NOTE

REMEMBER

DERIVATION

STRATEGY

Formulas sit on graph paper • coloured words are the key terms

Planetary Motion and the Law of Gravitation

All objects in the universe **attract** one another. This pull is called the **force of gravitation**. **Galileo** was the first to see that all bodies fall towards the Earth with a **constant acceleration**. Their masses make no difference. **Kepler** believed that the heavenly bodies move in a **mathematical order**. He thought this order was not like anything on the Earth.

1.1 Theory of Planetary Motion

From early times, people watched the **heavenly bodies**, such as the **Sun, the Moon, the stars and the planets**. They believed that all of these bodies go around the Earth. They thought the Earth was **fixed** at the **centre of the universe**.

i Geocentric Model

In this model, the **Earth is at the centre**. All the heavenly bodies move in orbits around it.

It is the **earliest recorded model**. **Ptolemy** gave it about **2000 years ago**.

ii Heliocentric Model

In this model, the **Sun is at the centre**. The planets revolve around the Sun.

It is a **more elegant model**. **Galileo** supported it after his experiments and studies of the Moon and the planets.

Tycho Brahe (1546–1601) observed the planets with his **naked eyes** for his whole life. He kept a careful record of what he saw. His assistant, **Johannes Kepler**, collected and studied this data. From it, Kepler found **three neat laws**. Today we call them **Kepler's laws**.

Tycho Brahe's data → Kepler's study → Kepler's three laws

1.2 Kepler's First Law (Law of Orbits)

Kepler's laws are **three laws** of science. They describe how the **planets move around the Sun**.

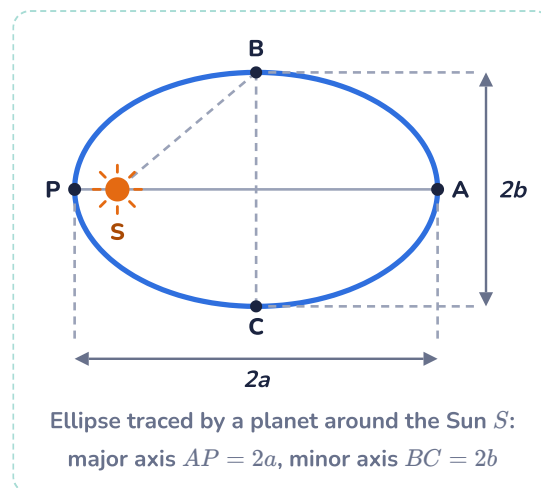
LAW

Law of Orbits

Kepler's first law: Every planet moves around the Sun in an **elliptical orbit**. The Sun is at **one focus** of the ellipse.

This law tells us that the **Sun–Earth distance** does not stay the same. It **keeps changing** while the Earth moves along its orbit.

In the figure, a planet traces an ellipse around the Sun S . The planet comes **closest** to the Sun at point P . It goes **farthest** from the Sun at point A .



P Perihelion

The point of the orbit where the planet is **nearest to the Sun**.

A Aphelion

The point of the orbit where the planet is **farthest from the Sun**.

Length of the **major axis** = Sun–planet distance at perihelion + Sun–planet distance at aphelion

$$AP = SP + SA$$

Length of the **major axis**

$$AP = 2a$$

Semi-major axis, which is half of the distance AP

$$a = \frac{AP}{2}$$

1.3 Kepler's Second Law (Law of Areas)

LAW

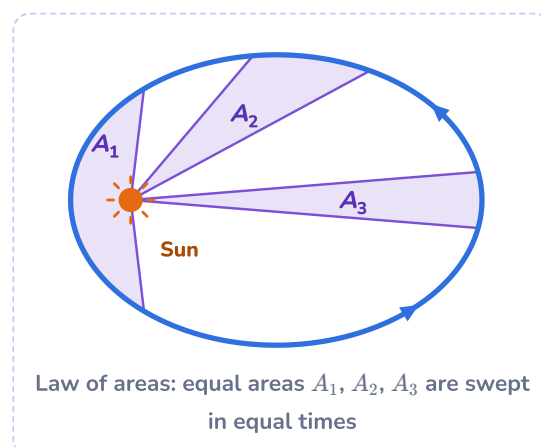
Law of Areas

Kepler's second law: The line from the Sun to a planet, called the **radius vector**, sweeps out **equal areas in equal intervals of time**. In other words, the planet's **areal velocity** stays **constant**. For this, the **speed of the planet changes** along its orbit.

In the figure, a planet moves on an elliptical orbit around the Sun. The radius vector sweeps out the areas A_1 , A_2 and A_3 in **equal intervals of time**.

So, by Kepler's second law:

$$A_1 = A_2 = A_3$$

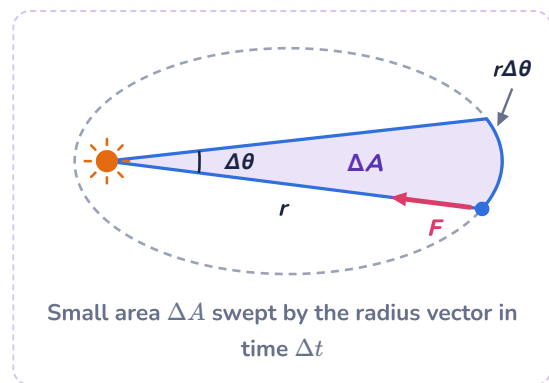


NOTE

Areal velocity is the **area** that the radius vector sweeps out **per unit time**.

Kepler's second law is the same as the **law of conservation of angular momentum**. To see this, take a small area ΔA swept in a small time interval Δt .

Let $\vec{r}$ be the position of the planet from the Sun. In time Δt , the radius vector turns through a small angle $\Delta\theta$. The force $\vec{F}$ on the planet points towards the Sun.



DERIVATION

Areal Velocity of a Planet Is Constant

- 1 The thin sector has radius r and arc length $r \Delta\theta$. Its area is:

$$\Delta A = \frac{1}{2} r (r \Delta\theta)$$

- 2 Divide both sides by the time interval Δt :

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$$

- 3 Now let $\Delta t \rightarrow 0$. The ratio $\frac{\Delta\theta}{\Delta t}$ becomes the **angular velocity** ω :

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega \quad \left[\because \omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} \right] \quad \dots (i)$$

- 4 The angular momentum of the planet (mass m) about the Sun at this instant is:

$$L = m r^2 \omega \quad \dots (ii)$$

- 5 Combine Eqs. (i) and (ii):

$$\frac{dA}{dt} = \frac{L}{2m} \quad \text{or} \quad L = 2m \times \frac{dA}{dt} \quad \dots (iii)$$

- 6 The gravitational force on the planet acts along the line through the Sun. So the angle between $\vec{r}$ and $\vec{F}$ is 180° , and the torque is zero:

$$\vec{\tau}_{\text{ext}} = \vec{r} \times \vec{F} = rF \sin 180^\circ \hat{n} = 0 \quad \therefore \tau_{\text{ext}} = 0$$

- 7 There is no external torque. So the angular momentum does not change with time:

$$L = \text{constant} \quad \left[\because \frac{dL}{dt} = \tau_{\text{ext}} = 0 \right] \quad \dots (iv)$$

- 8 From Eqs. (iii) and (iv), the areal velocity stays constant. This proves Kepler's second law.

$$\text{Areal velocity, } \frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

AREAL VELOCITY OF A PLANET

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

where, L = angular momentum of the planet about the Sun, m = mass of the planet

1.4 Kepler's Third Law (Law of Periods)

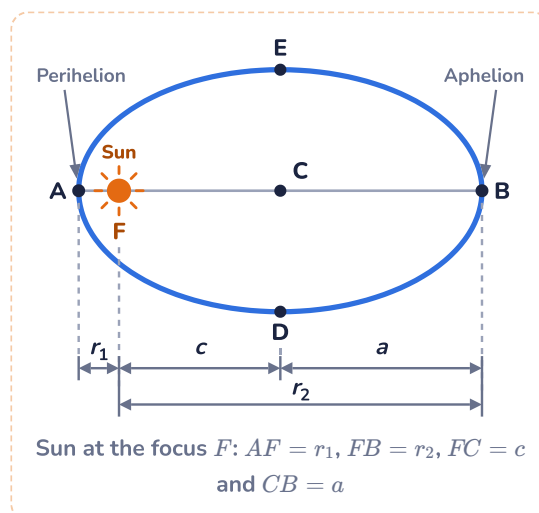
LAW

Law of Periods

Kepler's third law: For every planet, the **square of its time period** of revolution around the Sun is **directly proportional** to the **cube of the semi-major axis** of its orbit.

Let T be the **time period** of revolution of a planet. In the figure, the Sun is at the focus F of the orbit. Here, r_1 is the **shortest distance** of the planet from the Sun. r_2 is its **longest distance** from the Sun.

$$T^2 \propto a^3 \quad \dots (i)$$



DERIVATION

Law of Periods in Terms of r_1 and r_2

- 1 Kepler's third law for a planet with time period T and semi-major axis a is:

$$T^2 \propto a^3 \quad \dots (i)$$

- 2 From the figure, the major axis AB is made of the two parts AF and FB :

$$AB = AF + FB \quad \Rightarrow \quad 2a = r_1 + r_2$$

3 So the semi-major axis of the orbit is:

$$a = \frac{r_1 + r_2}{2}$$

4 Put this value of a in Eq. (i):

$$T^2 \propto \left(\frac{r_1 + r_2}{2} \right)^3$$

KEPLER'S THIRD LAW

$$T^2 \propto a^3 \qquad T^2 \propto \left(\frac{r_1 + r_2}{2} \right)^3$$

where, a = semi-major axis, r_1 = shortest distance of the planet from the Sun (perihelion), r_2 = longest distance of the planet from the Sun (aphelion)

NOTE

Kepler's laws are true **not only for the solar system**. They also hold for **artificial satellites** and for **moons** that go around the planets.

• KEPLER'S LAW OF PERIODS FOR THE SOLAR SYSTEM

In the table, the last column T^2/r^3 is **nearly the same** for every planet. This confirms the law.

PLANET	SEMI-MAJOR AXIS r (10^{10} M)	TIME PERIOD T (YEARS)	T^2/r^3 (10^{-34} $\text{y}^2 \text{m}^{-3}$)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	165.0	2.99
Pluto*	590	248.0	2.99

*Pluto is now classed as a **dwarf planet**. Its data still fits the law.

1.5 Universal Law of Gravitation

LAW

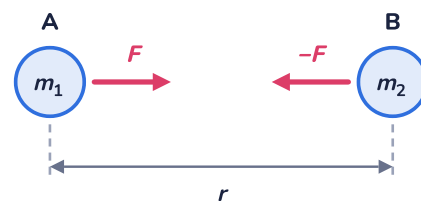
Newton's Law of Gravitation

Newton's law of gravitation: Every body in the universe **attracts** every other body with a force. This force is **directly proportional to the product of their masses** ($m_1 m_2$). It is **inversely proportional** to the **square of the distance** r between their centres. The force acts **along the line joining the centres** of the two bodies.

Take two bodies of masses m_1 and m_2 . The distance between their centres is r . Let F be the magnitude of the **gravitational force** between them. Newton's law says:

$$F \propto m_1 m_2 \quad \dots (i) \quad F \propto \frac{1}{r^2} \quad \dots (ii)$$

Combining (i) and (ii), $\therefore F \propto \frac{m_1 m_2}{r^2}$



Two bodies pull each other with equal and opposite forces

NEWTON'S LAW OF GRAVITATION

$$F = G \frac{m_1 m_2}{r^2} \quad \dots (iii)$$

where, G = constant of proportionality. It is called the **gravitational constant** or the **universal gravitational constant**.

Value of G in the **SI system**

$$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$$

Value of G in the **CGS system**

$$G = 6.67 \times 10^{-8} \text{ dyne cm}^2 \text{ g}^{-2}$$

DERIVATION

Dimensional Formula of G

1 Rearrange Eq. (iii) to get G :

$$G = \frac{F r^2}{m_1 m_2}$$

2 Write the dimensions of force, distance and mass:

$$G = \frac{[\text{MLT}^{-2}][\text{L}^2]}{[\text{M}^2]} = [\text{MLT}^{-2}][\text{L}^2][\text{M}^{-2}]$$

3 So the dimensional formula of the universal gravitational constant is:

$$G = [\text{M}^{-1} \text{L}^3 \text{T}^{-2}]$$

CONCEPT

Physical Meaning of G

Let $m_1 = m_2 = 1$ unit and $r = 1$ unit. Then Eq. (iii) gives $F = \frac{G \times 1 \times 1}{(1)^2}$, so $F = G$.

So G is **numerically equal** to the force of attraction between two bodies of **unit mass** each, kept a **unit distance** apart.

• IMPORTANT POINTS ABOUT GRAVITATIONAL FORCE

1 Always Attractive

Gravitational force **always attracts**. Electric and magnetic forces can **attract or repel**.

2 Medium Has No Effect

It does **not depend on the medium** between the bodies. Electric and magnetic forces **do depend** on the medium.

3 Conservative Force

Gravitational force is a **conservative force**.

4 G Never Changes

The value of G does **not depend** on the nature or the size of the masses.

5 Point Mass outside a Shell

A uniform **hollow spherical shell** pulls a point mass **outside** it as if all its mass were at its **centre**. So gravity has **spherical symmetry**.

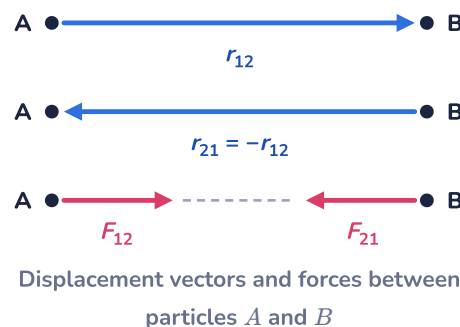
6 Point Mass inside a Shell

A uniform hollow spherical shell puts **zero force** on a point mass kept **inside** it.

1.6 Vector Form of Newton's Law of Gravitation

Take two particles A and B with masses m_1 and m_2 . Their displacement vectors and the forces between them are named like this:

- ◆ $\vec{r}_{12}$ = displacement vector **from A to B**
- ◆ $\vec{r}_{21}$ = displacement vector **from B to A** , so $\vec{r}_{21} = -\vec{r}_{12}$
- ◆ $\vec{F}_{12}$ = gravitational force **on A due to B**
- ◆ $\vec{F}_{21}$ = gravitational force **on B due to A**



DERIVATION

Vector Form and Central Force

- 1 Let $\hat{r}_{21}$ be the unit vector that points **towards A** . In vector form, the force on A due to B is:

$$\vec{F}_{12} = -G \frac{m_1 m_2}{r_{21}^2} \hat{r}_{21} \quad \dots (i)$$

2 The **minus sign** means that $\vec{F}_{12}$ is **opposite** to $\hat{r}_{21}$. So gravity pulls the particles together. It is an **attractive** force.

3 In the same way, let $\hat{r}_{12}$ be the unit vector that points **towards** B . The force on B due to A is:

$$\vec{F}_{21} = -G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12}$$

4 But $\hat{r}_{21} = -\hat{r}_{12}$ and $r_{21}^2 = r_{12}^2$. Put these in:

$$\vec{F}_{21} = G \frac{m_1 m_2}{r_{21}^2} \hat{r}_{21} \quad \dots (ii)$$

5 Compare Eqs. (i) and (ii). Both forces point **towards the centres** of the particles. So gravitational force is a **central force**.

$$\vec{F}_{12} = -\vec{F}_{21}$$

VECTOR FORM OF THE LAW

$$\vec{F}_{12} = -G \frac{m_1 m_2}{r_{21}^2} \hat{r}_{21}$$

where, $\hat{r}_{21}$ = unit vector pointing towards A . The negative sign means the force is attractive.

KEY $\vec{F}_{12} = -\vec{F}_{21}$. Gravity acts along the line of centres, so it is a **central force**

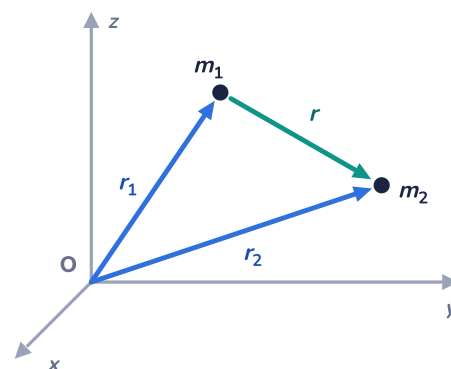
1.7 Principle of Superposition

PRINCIPLE

Let $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ be the forces due to the masses $m_1, m_2, m_3, \dots, m_n$. Each force follows the universal law of gravitation. The **principle of superposition** says that each of these forces acts **independently**. The other bodies have **no effect** on it.

The force on m_1 due to m_2 acts along the vector $\vec{r}$. Here $\vec{r}_1$ and $\vec{r}_2$ are the positions of the two particles, measured from the origin O .

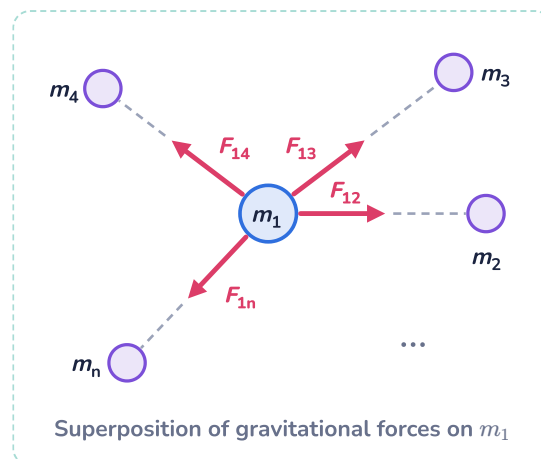
$$\vec{r} = \vec{r}_2 - \vec{r}_1$$



Force on m_1 due to m_2 acts along $\vec{r} = \vec{r}_2 - \vec{r}_1$

Now let many masses $m_2, m_3, m_4, \dots, m_n$ pull on m_1 . Each force $\vec{F}_{12}, \vec{F}_{13}, \dots, \vec{F}_{1n}$ points from m_1 **towards the other mass**.

The **resultant force** on m_1 is the **vector sum** of all these forces.



DERIVATION

Resultant Gravitational Force on m_1

- 1 The resultant force $\vec{F}$ on m_1 is the **vector sum** of all the separate forces:

$$\vec{F} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \dots + \vec{F}_{1n}$$

- 2 Write each force using the vector form of Newton's law of gravitation:

$$\vec{F} = -G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{21} - G \frac{m_1 m_3}{r_{13}^2} \hat{r}_{31} - \dots - G \frac{m_1 m_n}{r_{1n}^2} \hat{r}_{n1}$$

- 3 Take $-Gm_1$ common. The resultant force is:

$$\vec{F} = -Gm_1 \left[\frac{m_2}{r_{12}^2} \hat{r}_{21} + \frac{m_3}{r_{13}^2} \hat{r}_{31} + \dots + \frac{m_n}{r_{1n}^2} \hat{r}_{n1} \right]$$

RESULTANT FORCE BY SUPERPOSITION

$$\vec{F} = -Gm_1 \left[\frac{m_2}{r_{12}^2} \hat{r}_{21} + \frac{m_3}{r_{13}^2} \hat{r}_{31} + \dots + \frac{m_n}{r_{1n}^2} \hat{r}_{n1} \right]$$

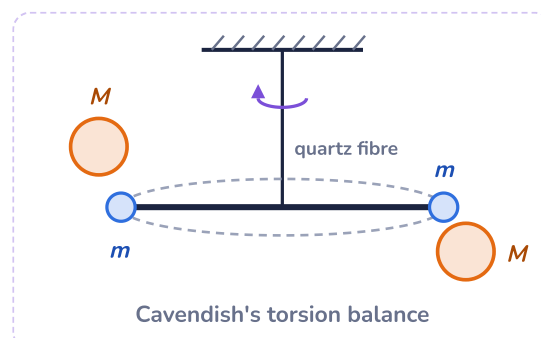
It gives the total pull on m_1 from all the other masses. Each force acts on its own, free of the others.

1.8 Gravitational Constant (Cavendish's Experiment)

The English scientist **Henry Cavendish** was the **first to measure** the value of G by experiment. He did this in **1798**.

Two **small lead spheres** (mass m each) sit at the ends of a **rod of length L** . A **fine quartz fibre** holds the rod at its middle. This set-up is a **torsion balance**.

Two **large lead spheres** (mass M each) are brought near the small ones, on **opposite sides** of the rod.



DERIVATION

Measuring G with the Torsion Balance

- 1 Each small sphere is pulled towards the big sphere next to it. Let R be the distance between their centres. The force on each small sphere is:

$$F = \frac{GMm}{R^2}$$

- 2 The two forces are equal and opposite. So the **net force on the rod is zero**. But they produce a **gravitational torque** on it:

$$\text{Gravitational torque} = \text{gravitational force} \times \text{length of the bar} = \frac{GMm}{R^2} \times L$$

- 3 This torque turns the rod through an angle θ , with the fibre as the axis. So the fibre gets **twisted through θ** . Let τ be the restoring torque **per unit twist** of the fibre:

$$\text{Restoring torque} = \tau \theta$$

- 4 In equilibrium, the restoring torque equals the gravitational torque:

$$\tau \theta = \frac{GMm}{R^2} \times L$$

- 5 Every quantity on the right side is known from the experiment. So G can be found:

$$G = \frac{R^2 \tau \theta}{M m L}$$

GRAVITATIONAL CONSTANT (CAVENDISH)

$$G = \frac{R^2 \tau \theta}{M m L}$$

Later experiments have measured G more accurately. Its accepted value is $G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$.

NOTE

G is a **universal constant**. Its value does **not change** with the medium between the bodies. It also does not change with **temperature, pressure** and so on.

GOOD TO KNOW

Newton's Principia

Kepler had stated his third law by **1619**. Newton announced the universal law of gravitation behind it about **seventy years later**, in **1687**. It came out in his great book ***Philosophiae Naturalis Principia Mathematica***, usually called the **Principia**.

2.1 Acceleration due to Gravity of the Earth

DEFINITION

The acceleration of a body that **falls freely** under gravity is called the **acceleration due to gravity**. It is equal to the **force of gravity on a unit mass** kept on or near the Earth's surface. Its symbol is g . On the Earth's surface, $g = 9.8 \text{ m/s}^2$.

Value of g for all bodies at one place, whatever their masses

Same ✓

Value of g at different places on the Earth's surface

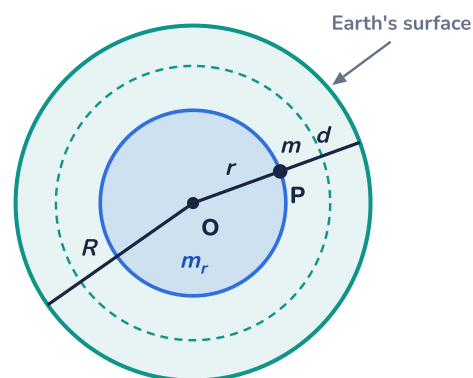
Differs

Value of g with **altitude** and **depth**

Varies

For a point **outside the Earth**, the Earth pulls as if its **whole mass** were at its **centre**.

Imagine the Earth as a set of **concentric shells**. A point mass m is at P , a distance r from the centre O . Point P lies **outside** the sphere of radius r . It lies **inside** every shell of radius larger than r .



Finding g : the Earth as concentric shells

DERIVATION

 g on the Surface of the Earth

- The inner sphere has radius r and mass m_r . It pulls m as if the mass m_r were at the **centre**. The shells of radius larger than r put **no force** on m . So the force on m at P is:

$$F = \frac{G m (m_r)}{r^2} \quad \dots (i)$$

- Write the masses using the density ρ of the Earth. Here M is the mass and R the radius of the Earth:

$$M = \frac{4\pi}{3} R^3 \rho \quad m_r = \frac{4\pi}{3} \rho r^3$$

3 Put the value of m_r in Eq. (i):

$$F = \frac{Gm}{r^2} \cdot \frac{4\pi}{3}\rho r^3 = Gm \frac{M}{R^3} r$$

4 On the surface of the Earth, $r = R$:

$$F = G \frac{Mm}{R^2}$$

5 By Newton's second law, $F = mg$. So:

$$g = \frac{F}{m} = \frac{GM}{R^2}$$

ACCELERATION DUE TO GRAVITY

$$g = \frac{GM}{R^2}$$

where, M = mass of the Earth, R = radius of the Earth. With $M \approx 6 \times 10^{24}$ kg and $R = 6.4 \times 10^6$ m, we get $g = 9.8 \text{ m/s}^2$.

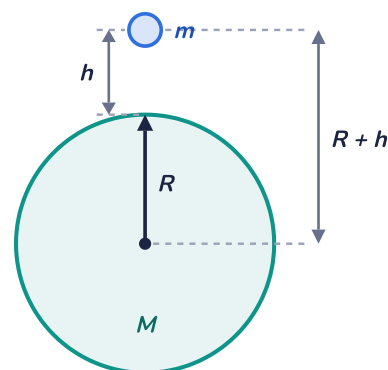
NOTE

- ◆ On the surface of the **Moon**, g is $\frac{1}{6}$ of its value on the surface of the Earth.
- ◆ g does **not depend** on the mass, shape or size of the body. It depends on the **mass and radius of the Earth**.

2.2 Variation of g with Altitude

A body of mass m lies on the surface of the Earth. The Earth has mass M and radius R .

Now the body is taken to a **height h** above the surface. Let the acceleration due to gravity there be g_h . The centre of the body is now at a distance $(R + h)$ from the centre of the Earth.



Body at a height h above the Earth's surface

DERIVATION

g at a Height h

1 At the surface of the Earth:

$$g = \frac{GM}{R^2} \quad \dots (i)$$

2 At height h , the body is at a distance $(R + h)$ from the centre of the Earth:

$$g_h = \frac{GM}{(R + h)^2} \quad \dots (ii)$$

3 Divide Eq. (ii) by Eq. (i):

$$\frac{g_h}{g} = \frac{GM}{(R + h)^2} \times \frac{R^2}{GM} = \frac{R^2}{(R + h)^2} \quad \dots (iii)$$

4 Write $(R + h)$ as $R \left(1 + \frac{h}{R}\right)$:

$$\frac{g_h}{g} = \frac{R^2}{R^2 \left(1 + \frac{h}{R}\right)^2} = \left(1 + \frac{h}{R}\right)^{-2} \Rightarrow g_h = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$$

5 If $h \ll R$, then h/R is very small. Expand by the **binomial theorem**. Drop the squares and higher powers of h/R :

$$\frac{g_h}{g} = 1 - \frac{2h}{R}$$

6 So the acceleration due to gravity at height h is:

$$g_h = g \left(1 - \frac{2h}{R}\right) \quad \dots (iv) \quad \text{or} \quad g - g_h = \frac{2h}{R} g \quad \dots (v)$$

G AT HEIGHT H

$$g_h = \frac{g}{\left(1 + \frac{h}{R}\right)^2} \quad g_h = g \left(1 - \frac{2h}{R}\right) \quad (h \ll R)$$

where, g = acceleration due to gravity at the surface, R = radius of the Earth, h = height above the surface

DECREASE IN g

$$g - g_h = \frac{2hg}{R}$$

FRACTIONAL DECREASE

$$\frac{g - g_h}{g} = \frac{2h}{R}$$

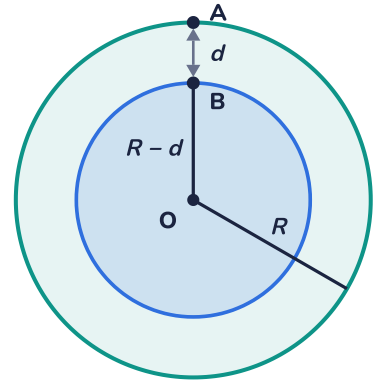
$$\left(\frac{g - g_h}{g} \right) \times 100 = \frac{2h}{R} \times 100\%$$

KEY From Eq. (v), $g - g_h \propto h$. So g decreases as the height increases

2.3 Variation of g with Depth

Take the Earth as a **homogeneous sphere**, that is, a sphere of **uniform density**. Let ρ be its **mean density**. A body on the surface has acceleration due to gravity g .

Now take the body to a **depth d** below the surface. The gravity on it now comes **only from the inner solid sphere** of radius $(R - d)$.



Body at a depth d below the Earth's surface

DERIVATION

g at a Depth d

- 1 At the surface, $g = \frac{GM}{R^2}$. The mass of the Earth is $M = \frac{4}{3}\pi R^3 \rho$. So:

$$g = \frac{G \times \frac{4}{3}\pi R^3 \rho}{R^2} = \frac{4}{3}\pi G R \rho \quad \dots (i)$$

- 2 At depth d , only the inner solid sphere pulls the body. It has radius $(R - d)$ and mass M' :

$$g_d = \frac{GM'}{(R - d)^2}$$

- 3 The mass of the inner solid sphere is $M' = \frac{4}{3}\pi (R - d)^3 \rho$. Put it in:

$$g_d = \frac{G}{(R - d)^2} \times \frac{4}{3}\pi (R - d)^3 \rho = \frac{4}{3}\pi G (R - d) \rho \quad \dots (ii)$$

- 4 Divide Eq. (ii) by Eq. (i):

$$\frac{g_d}{g} = \frac{\frac{4}{3}\pi G (R - d) \rho}{\frac{4}{3}\pi G R \rho} = \frac{R - d}{R} = 1 - \frac{d}{R}$$

- 5 So the acceleration due to gravity at depth d is given below. Here $(g - g_d)$ is the **decrease** in the value of g .

$$g_d = g \left(1 - \frac{d}{R} \right) \quad \text{or} \quad g - g_d = \frac{d}{R} g$$

G AT DEPTH D

$$g_d = g \left(1 - \frac{d}{R} \right)$$

where, g = acceleration due to gravity at the surface, R = radius of the Earth, d = depth below the surface

DECREASE IN G

$$g - g_d = \frac{g d}{R}$$

FRACTIONAL DECREASE

$$\frac{g - g_d}{g} = \frac{d}{R}$$

PERCENTAGE DECREASE

$$\left(\frac{g - g_d}{g} \right) \times 100 = \frac{d}{R} \times 100\%$$

KEY $(g - g_d) \propto d$. So g decreases as we go down below the Earth's surface

REMEMBER

At the Centre of the Earth

At the centre O of the Earth, $d = R$. So $g_d = g \left(1 - \frac{R}{R} \right) = 0$. So g becomes **zero** at the Earth's centre.

So the **weight of a body is zero** at the centre of the Earth. But its **mass stays the same**.

NOTE

Effect of the Earth's Rotation

The **rotation of the Earth** also changes g . Let a body of mass m be at a place of **latitude** λ . Let ω be the angular speed of the Earth. Then the body has an **apparent acceleration due to gravity** given by:

$$g' = g - \omega^2 R \cos^2 \lambda$$

2.4 Intensity of Gravitational Field

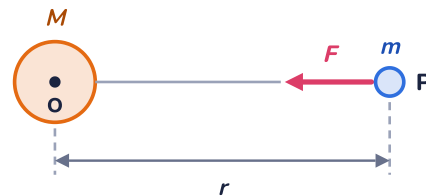
DEFINITION

The **intensity of the gravitational field** at a point is the **force on a unit mass** placed at that point.

A body of mass M has a **gravitational field** around it. To find the field intensity at a point P , place a **test mass** m at P .

The test mass must be **very small**. Then it does **not change** the field at all. Let $\vec{F}$ be the force on m at P . Then the intensity is:

$$I = \frac{F}{m} = \frac{GMm}{r^2 m}$$



Test mass m in the field of a body of mass M

INTENSITY OF GRAVITATIONAL FIELD

$$I = \frac{GM}{r^2}$$

where, r = distance of the point P from the centre of the body that produces the field

If $r = \infty$, then $I = 0$. The field intensity is zero **only at an infinite distance** from the body.

$$I = 0$$

For a unit mass on the **surface of the Earth**, the force on it is its weight w . So $I = \frac{w}{m} = \frac{mg}{m}$

$$I = g$$

2.5 Gravitational Potential

DEFINITION

The **gravitational potential** at a point in a gravitational field is the **work done** to move a **unit mass** from **infinity** up to that point, with **no acceleration**.

$$V = -\frac{W}{m} = -\int \frac{\vec{F} \cdot d\vec{r}}{m} = -\int \vec{I} \cdot d\vec{r} \quad \left[\text{as } \frac{F}{m} = I \right]$$

Here, W is the work done **by the gravitational force** as a body of mass m comes from infinity to the point. I is the intensity of the field. From this relation:

FIELD FROM POTENTIAL

$$I = -\frac{dV}{dr}$$

The **negative gradient of the potential** gives the field intensity. Potential is a scalar function of position. Its rate of change with distance gives the intensity.

The **negative sign** shows that the field intensity points in the direction in which the **potential decreases**.

Nature

Scalar quantity.

Units

SI: J kg^{-1} (joule/kg). CGS: erg g^{-1} (erg/g).

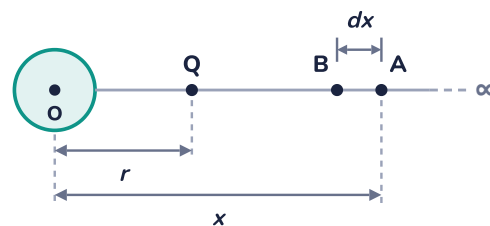
Dimensions

$$V = \frac{W}{m} = \frac{[\text{ML}^2\text{T}^{-2}]}{[\text{M}]} \\ = [\text{M}^0\text{L}^2\text{T}^{-2}]$$

• EXPRESSION FOR GRAVITATIONAL POTENTIAL

Take the Earth as a **perfect sphere** of radius R and mass M . Its mass acts as if it were all at its **centre O** .

We find the potential at a point Q , where $OQ = r$ and $r > R$. Take two points A and B on the same line, with $OA = x$ and $AB = dx$.



Finding the gravitational potential at Q

DERIVATION

Gravitational Potential at a Point

- At point A , the gravitational pull on a body of **unit mass** is:

$$F = \frac{GM \times 1}{x^2} = \frac{GM}{x^2}$$

- Move the unit mass **without acceleration** through the small distance $AB = dx$. The small work done is:

$$dW = F dx = \frac{GM}{x^2} dx$$

- The total work done in bringing the body from **infinity to the point Q** is:

$$W = \int_{\infty}^r \frac{GM}{x^2} dx = - \left[\frac{GM}{x} \right]_{\infty}^r$$

- Put in the limits:

$$W = -GM \left[\frac{1}{r} - \frac{1}{\infty} \right] = \frac{-GM}{r}$$

- This is the work done on a unit mass, so $V = W$. The gravitational potential at Q is:

$$V = \frac{-GM}{r} \quad \dots (i)$$

GRAVITATIONAL POTENTIAL

$$V = -\frac{GM}{r}$$

where, V = gravitational potential at the point Q , r = distance of Q from the Earth's centre

i At Infinity

$$r = \infty \Rightarrow V_Q = 0$$

The gravitational potential is **maximum (zero)** at infinity.

ii At the Surface

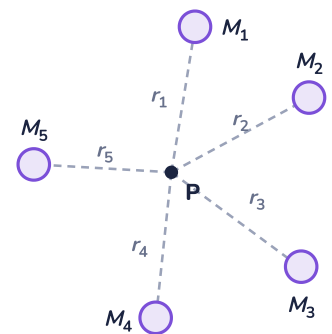
$$r = R \Rightarrow V_Q = -\frac{GM}{R}$$

This is the potential on the **surface of the Earth**.

NOTE

The potential due to **many particles** is the **scalar sum** of their separate potentials. So the potentials simply add, like ordinary numbers.

The formula below gives the potential at a point P due to masses $M_1, M_2, M_3, \dots$ kept at distances $r_1, r_2, r_3, \dots$ from it.



Superposition of potentials at P

POTENTIAL DUE TO MANY MASSES

$$V = V_1 + V_2 + V_3 + \dots = \frac{-GM_1}{r_1} - \frac{GM_2}{r_2} - \dots = -G \sum_{i=1}^{i=n} \frac{M_i}{r_i}$$

2.6 Gravitational Potential Energy

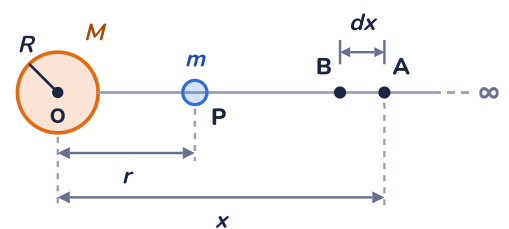
DEFINITION

The **work done against the gravitational force** in bringing a body from **infinity** to a point is called the **gravitational potential energy** of the body at that point.

A body of mass m is at a point P in the gravitational field of a body of mass M . Let r be the distance of P from the centre O . The total work done **against gravity** in moving the body from infinity to the distance r is:

$$W = -\frac{GmM}{r}$$

This work is **equal to the gravitational potential energy** U of the mass m .



Mass m at a distance r from the centre O

$$U = -\frac{GMm}{r} \qquad U = \left(-\frac{GM}{r}\right) \times m$$

where, r = distance of the body of mass m from the centre of the body of mass M

Potential energy = Mass $\times$ Gravitational potential

• SPECIAL CASES

i System of Three Particles

Take three particles of masses m_1 , m_2 and m_3 in an **isolated system**. By the **superposition principle**, the total gravitational potential energy of the system is:

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_2m_3}{r_{23}} + \frac{Gm_1m_3}{r_{13}}\right)$$

DERIVATION

(ii) Body Raised to a Height h

A body of mass m is lifted from the **Earth's surface** to a **height h** above it.

- 1 The change in potential energy, or the **work done against gravity**, in moving from r_1 to r_2 is:

$$W = \Delta U = GMm \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

- 2 Take $r_1 = R$ (the surface) and $r_2 = R + h$ (at height h):

$$\Delta U = GMm \left(\frac{1}{R} - \frac{1}{R+h} \right)$$

- 3 Simplify, and use $\frac{GM}{R^2} = g$:

$$\Delta U = \frac{GMmh}{R^2 \left(1 + \frac{h}{R}\right)} = \frac{mgh}{1 + \frac{h}{R}}$$

- 4 Expand $(1 + h/R)^{-1}$ by the binomial theorem:

$$\Delta U = mgh \left(1 + \frac{h}{R}\right)^{-1} = mgh \left(1 - \frac{h}{R}\right)$$

- 5 Since $h \ll R$, the term h/R is very small. We can neglect it:

$$\Delta U = mgh$$

iii Height Equal to the Earth's Radius

Now lift the body to a height $h = R$, that is, one Earth radius above the surface. The work done is:

$$W = \frac{mgR}{1 + R/R} = \frac{1}{2} mgR$$

03 PART 3 Escape Speed and Satellites

3.1 Escape Speed

DEFINITION

The **escape speed** is the **minimum speed** with which a body must be thrown **vertically upwards** from the Earth's surface so that it **just escapes** the Earth's gravitational field. The body then **never returns** on its own. The same idea holds for any other planet.

i Usual Projectile ($v < v_e$)

A body thrown upwards normally **slows down**. It stops for a moment and then **falls back** to the Earth.

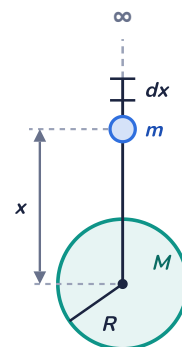
ii Minimum Initial Speed ($v = v_e$)

The body **keeps moving upwards** forever. It comes to rest **only at infinity**.

A body of mass m is at a distance x from the centre of the Earth. Let M be the mass and R the radius of the Earth.

By Newton's law of gravitation, the Earth pulls the body with a

gravitational force $F = \frac{GMm}{x^2}$.



DERIVATION

Escape Speed

1 At a distance x , the gravitational pull on the body is:

$$F = G \frac{Mm}{x^2}$$

2 The work done to move the body up through a small distance dx is:

$$dW = F dx = \frac{GMm}{x^2} dx$$

3 Find the total work done to take the body from the surface ($x = R$) to $x = \infty$. Here GMm is a constant:

$$W = \int dW = GMm \int_R^\infty x^{-2} dx = -GMm \left[\frac{1}{x} \right]_R^\infty$$

4 Put in the limits, using $\frac{1}{\infty} = 0$:

$$W = -GMm \left[\frac{1}{\infty} - \frac{1}{R} \right] = \frac{GMm}{R}$$

5 Let v_e be the escape speed. The kinetic energy given to the body must be equal to this work:

$$\frac{1}{2} mv_e^2 = \frac{GMm}{R}$$

6 Solve for v_e :

$$v_e^2 = \frac{2GM}{R} \Rightarrow v_e = \sqrt{\frac{2GM}{R}} \quad \dots (i)$$

KEY

Escape speed depends on the mass and radius of the planet. It does not depend on the mass of the body

In Terms of g

Since $g = \frac{GM}{R^2}$, we have $GM = gR^2$. So

$$v_e = \sqrt{\frac{2gR^2}{R}}:$$

$$v_e = \sqrt{2gR} \quad \dots (ii)$$

In Terms of Mean Density ρ

Since $M = \frac{4}{3}\pi R^3 \rho$, we get $v_e = \sqrt{\frac{2G}{R} \times \frac{4}{3}\pi R^3 \rho}$:

$$v_e = R\sqrt{\frac{8}{3}\pi G\rho} \quad \dots (iii)$$

ESCAPE SPEED (THREE FORMS)

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} = R\sqrt{\frac{8}{3}\pi G\rho}$$

where, R = radius of the Earth (planet), g = acceleration due to gravity at its surface, ρ = mean density of the Earth (planet)

3.2 Escape Speed from Conservation of Energy

Suppose the object reaches **infinity** with a speed v_f . The total energy of an object is its **potential energy plus its kinetic energy**.

DERIVATION

Escape Speed Using Energy Conservation

- 1 Let W_1 be the gravitational potential energy of the object at infinity. Its total energy at infinity is:

$$E(\infty) = W_1 + \frac{mv_f^2}{2} \quad \dots (i)$$

- 2 Suppose the object was thrown with speed v_i from a point at a distance $(h + R)$ from the centre of the Earth. Its energy at the start is:

$$E(h + R) = \frac{1}{2}mv_i^2 - \frac{GmM}{h + R} + W_1 \quad \dots (ii)$$

- 3 By the principle of conservation of energy, Eqs. (i) and (ii) are equal:

$$\frac{mv_i^2}{2} - \frac{GmM}{(h + R)} = \frac{mv_f^2}{2}$$

- 4 The **minimum** speed is the one with which the object **just reaches infinity**. Then $\frac{mv_f^2}{2} = 0$:

$$\frac{1}{2}m(v_i^2)_{\min} = \frac{GmM}{h + R}$$

- 5 For an object thrown from the surface of the Earth, $h = 0$:

$$(v_i)_{\min} = \sqrt{\frac{2GM}{R}}$$

- 6 Use the relation $g = GM/R^2$:

$$\text{Minimum speed, } (v_i)_{\min} = \sqrt{2gR}$$

ESCAPE SPEED OF THE EARTH

$$(v_i)_{\min} = \sqrt{2gR}$$

With $g = 9.8 \text{ m/s}^2$ and $R = 6.4 \times 10^6 \text{ m}$, $(v_i)_{\min} = 11.2 \text{ km/s}$

NOTE

Orbital Velocity

The **orbital velocity** (v_o) of a satellite is the velocity a satellite must be given to place it in its orbit around the Earth. For an orbit close to the Earth's surface, $v_o = \sqrt{gR}$. So:

$$v_e = \sqrt{2} v_o$$

3.3 Earth Satellites

DEFINITION

A **satellite** is a body that keeps **revolving in an orbit** around a **much larger body**.

i Natural Satellite

e.g. The **Moon**

ii Artificial Satellite

e.g. **INSAT-1B**

NOTE

Condition for a Satellite

The **centre of the orbit** of the satellite must be at the **centre of the Earth**. In other words, the satellite must move along a **great circle** of the Earth.

DERIVATION

Orbital Speed and Time Period of a Satellite

- 1 A satellite of mass m moves in a circular orbit at a height h . It needs a **centripetal force** towards the centre:

$$F_c = \frac{mv^2}{R+h} \quad \dots (i)$$

- 2 The **gravitational force** of the Earth provides this centripetal force:

$$F_g = \frac{GmM}{(R+h)^2} \quad \dots (ii)$$

- 3 Equate Eqs. (i) and (ii). This shows that v **decreases as h increases**:

$$v^2 = \frac{GM}{R+h}$$

- 4 For an orbit very close to the surface, $h = 0$. Using $g = \frac{GM}{R^2}$:

$$v^2 = \frac{GM}{R} = gR$$

5

In one revolution, the satellite covers a distance $2\pi(R + h)$ at speed v . Using $v^2 = \frac{GM}{R + h}$:

$$T = \frac{2\pi(R + h)}{v} = \frac{2\pi(R + h)^{3/2}}{\sqrt{GM}} \quad \dots (iii)$$

6

Square both sides of Eq. (iii). The result is **Kepler's law of periods**:

$$T^2 = k(R + h)^3 \quad \left[\because k = \frac{4\pi^2}{GM} \right]$$

ORBITAL SPEED AND PERIOD

$$v = \sqrt{\frac{GM}{R + h}} \quad T^2 = k(R + h)^3, \quad k = \frac{4\pi^2}{GM}$$

where, R = radius of the Earth, h = height of the orbit above the surface, M = mass of the Earth

Orbit Just above the Earth's Surface

Here h is tiny compared with R , so we can **neglect** it. Put $h = 0$ and call the period T_0 :

$$T_0 = 2\pi\sqrt{R/g} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} \cong 85 \text{ min}$$

3.4 Energy of an Orbiting Satellite

A satellite going around a planet in its orbit has **two kinds of energy**. It has **potential energy** because of its **position** against the Earth's pull. It has **kinetic energy** because of its **orbital motion**.

DERIVATION

Energy of an Orbiting Satellite

1

Let m be the mass of the satellite and v its orbital velocity. Using $v^2 = \frac{GM}{r}$ with $r = R + h$, its kinetic energy is:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m \frac{GM}{r} \Rightarrow K = \frac{GMm}{2(R + h)} \quad \dots (i)$$

2

Its potential energy is mass $\times$ gravitational potential, that is, $U = mV$:

$$U = -\frac{GMm}{R + h} \quad \dots (ii)$$

3 The total mechanical energy is $E = K + U$:

$$E = \frac{GMm}{2(R+h)} - \frac{GMm}{(R+h)}$$

4 So the total energy of the orbiting satellite is:

$$E = -\frac{GMm}{2(R+h)} \quad \dots (iii)$$

ENERGY OF AN ORBITING SATELLITE

$$K = \frac{GMm}{2(R+h)} \quad U = -\frac{GMm}{R+h} \quad E = -\frac{GMm}{2(R+h)}$$

where, m = mass of the satellite, M = mass of the Earth, $(R+h)$ = radius r of the orbit

REMEMBER

Why Is E Always Negative?

A satellite always stays at a **finite distance** from the Earth. So its total energy can **never be positive or zero**.

The graph shows the **kinetic**, **potential** and **total** energies against the orbit radius r . It follows these relations:

(i) Kinetic energy

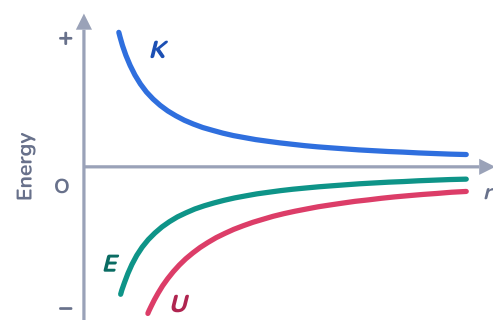
$$K = -(\text{total energy})$$

(ii) Potential energy

$$U = 2(\text{total energy})$$

(iii) Potential energy

$$U = -2(\text{kinetic energy})$$



Energies of a satellite against the orbit radius r

A satellite can also move in an **elliptical orbit** with semi-major axis a . Its total energy is then also **constant**:

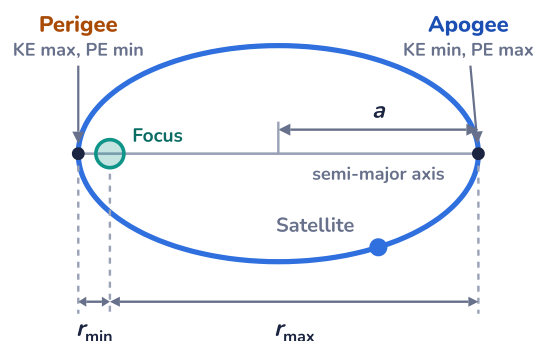
$$E = \frac{-GMm}{2a} = \text{constant}$$

Perigee ($r_{\min}$)

$$KE = \text{max} \cdot PE = \text{min}$$

Apogee ($r_{\max}$)

$$KE = \text{min} \cdot PE = \text{max}$$



Energy at different points of an elliptical orbit

3.5 Weightlessness

DEFINITION

Static or True Weight

The force with which the Earth pulls a body towards its **centre** is called the **weight** of the body. When the body is at rest with respect to the Earth, its weight equals the force of gravity on it. This weight is called its **static weight** or **true weight**.

A body in space can feel two gravitational pulls in opposite directions. At one particular point, these two pulls become **equal and opposite**. The net pull on the body is then **zero**. This point is called the **null point** or the **zero gravity region**. A body there is **weightless**.

The state of **weightlessness** is seen in these cases:

1 Free Fall

Bodies in **free fall** under gravity. **e.g.** a freely falling lift.

2 Orbiting Satellite

A **satellite revolving** in its orbit around the Earth.

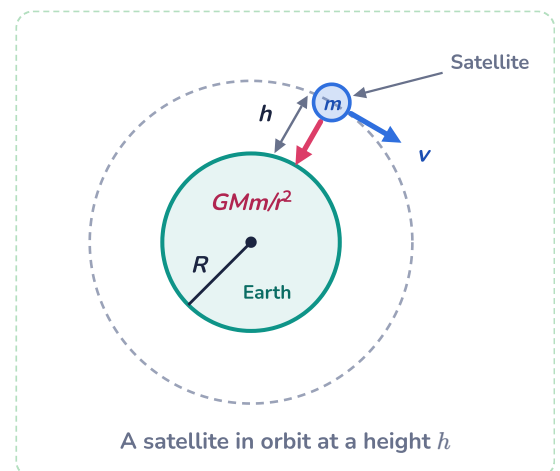
3 Null Points

Bodies at the **null points** in outer space.

WEIGHTLESSNESS IN A SATELLITE

Take a satellite that has **no gravity of its own**. Gravity alone keeps it moving in a circular orbit around the Earth.

The **acceleration of the satellite** is $\frac{GM}{r^2}$. It is directed towards the **Earth's centre**. Inside the satellite, a body of mass m rests on a surface.



DERIVATION

Apparent Weight in a Satellite ($N = 0$)

1 Two forces act on the body: the **pull of the Earth**, $\frac{GMm}{r^2}$, and the **reaction of the surface**, N .

2 Net force = mass $\times$ acceleration. The acceleration of the satellite is $a = \frac{GM}{r^2}$:

$$\frac{GMm}{r^2} - N = ma \quad \Rightarrow \quad \frac{GMm}{r^2} - N = m \left(\frac{GM}{r^2} \right)$$

3 So the surface puts **no force** on the body. Its **apparent weight is zero**:

$$N = 0$$

REMEMBER

Condition for Weightlessness

A body is weightless in a satellite only if the **mass of the satellite is negligible**. Then the satellite has no gravity of its own.

e.g. The **Moon** is a satellite of the Earth. But the Moon has its own large mass. It pulls a body kept on its surface. So the weight of the body on the Moon is **not zero**.

A **simple pendulum** experiment cannot be done in a weightless state, because

$$T = 2\pi\sqrt{\frac{L}{g}} = \infty \text{ as } g = 0$$

END OF CHAPTER 7 • NOTESSHALA.IN