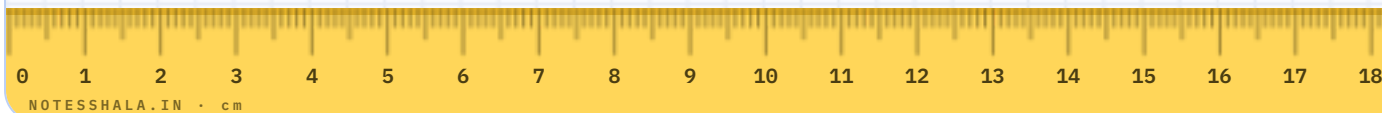


CHAPTER

04

Laws of Motion

Force, Inertia and Newton's First Law • Momentum, Second and Third Laws • Equilibrium, Friction and Circular Motion



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Motion is everywhere in nature. For large **(macroscopic) bodies**, we can see it easily. At the **atomic** and **nuclear** level, we cannot see motion, but it is still there. A motion can be as simple as a ball thrown up into the air. To study any motion, we must study the **forces** that control it.

1.1 Force

DEFINITION

Force is an external **push** or **pull** on a body. It **changes, or tries to change**, the body's state of rest or of uniform motion. It can also change the body's **direction of motion**.

SI Unit

newton (N)

Dimensional Formula

$[MLT^{-2}]$

A force on a body can produce **four types of effects**:

i Starts Motion

It can change the **state of rest** of a body into **motion**.

ii Changes Speed

It can change the **speed** of a body. The body then moves **faster** or **slower**.

iii Changes Direction

It can change the **direction of motion** of a body.

iv Changes Shape

It can change the **shape** of a body.

NOTE

Key Points About Force

- ◆ A **bar magnet** attracts an iron nail without touching it, from some **distance**. So forces such as **gravitational** and **magnetic** forces can act **without contact**.
- ◆ A force always has a **point of application**. So force is a **polar vector**.
- ◆ The **vector sum** of all the forces acting on a body is called the **net force**.

1.2 Aristotle's Fallacy

Think of a box sliding with **constant velocity** on a slab of ice. It is in **uniform motion** along a straight line. Is an **external force** needed to keep it moving? Aristotle's fallacy and Galileo's law of inertia answer this question.

CONCEPT

Aristotle's View (384 BC – 322 BC)

Aristotle was a Greek philosopher. He believed that a moving body needs an **external force** to **keep it moving**.

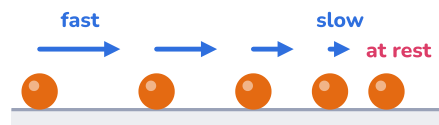
e.g. An arrow keeps flying after it leaves the bow. Aristotle's reason was that the **air behind it** pushes it forward all the time.

What **Aristotle's law** claims

Uniform motion needs an external force

Aristotle's idea comes from a simple fact. **Resistive forces** are always present, and they stop every motion. So a force seems necessary to **balance the resistive force**.

e.g. A **rolling ball** slows down and stops because of **friction**.



A ball on a horizontal floor: fast → slow → rest, due to friction

CONCEPT

Galileo Corrects Aristotle

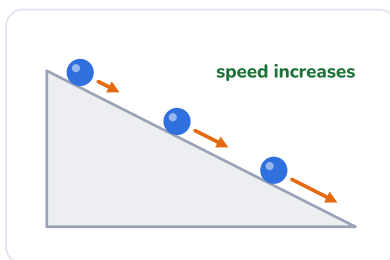
In nature, opposing forces are always present. Solids have **friction** and fluids have **viscous forces**. Nearly **two thousand years** after Aristotle, **Galileo** found the correct answer. An external force is required **only to cancel friction**. If there were no friction, a body in uniform motion would go on moving with **no external force** at all. In this way, Galileo proved the **law of inertia**.

1.3 Galileo's Single Inclined Plane

Galileo first studied objects moving on an **inclined plane**. He made three observations:

i Moving Down the Plane

The speed of the object **increases**.



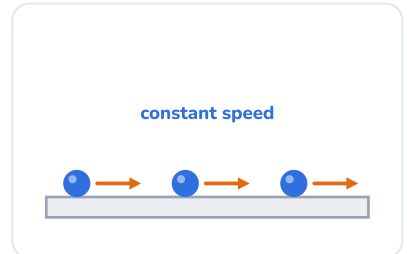
ii Moving Up the Plane

The speed **decreases**. The object **retards**.



iii Moving on a Horizontal Plane

The speed stays **constant**.



NOTE

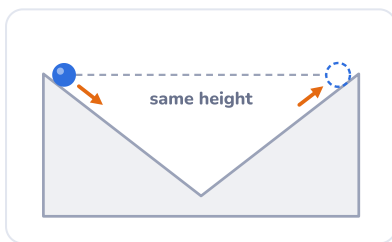
A level (**horizontal**) plane gives the object **no acceleration** and **no retardation**. So the object keeps a **constant speed** on it.

1.4 Galileo's Double Inclined Plane

Galileo saw that the bob of a swinging **pendulum** always rises to the **same height** on both sides of its mean position. He then did another experiment with a **double inclined plane**. It has two inclined planes **facing each other**.

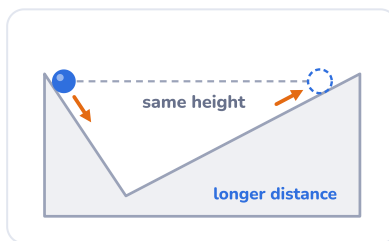
i Equal Slopes

The ball **rolls down** one plane and **climbs up** the other. Because of **friction**, it reaches **almost** the same height. With **no friction** (ideal case), the final height **equals** the initial height.



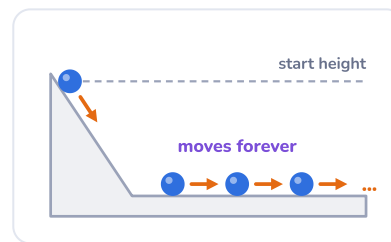
ii Second Slope Reduced

Now make the second plane **less steep**. The ball again rises to the **same height**, but it has to roll a **longer distance** to get there.



iii Second Slope Zero

The second plane is now **horizontal**. In the ideal case, the ball travels an **infinite distance**. It moves **forever with uniform velocity**. This is called **perpetual motion**.

**CONCEPT**

Conclusion: the Law of Inertia

In the ideal case, nothing slows the ball on a horizontal second plane. It goes on with **uniform velocity** without any push from outside. From this, Galileo arrived at the **law of inertia**. A body stays at rest, or keeps moving uniformly, until an **external force** acts on it.

1.5 Inertia

Galileo's experiments led to the **law of inertia**. When no external force acts, a body **at rest** and a body moving with **constant velocity** behave alike. The two states are **equivalent**.

DEFINITION

Inertia is the natural (inherent) property of a body due to which it **resists any change** in its state, whether that state is **rest** or **uniform motion along a straight line**. The word inertia means **resistance** to change. The idea of inertia was first given by **Galileo**.

• TYPES OF INERTIA

i Inertia of Rest

Inertia of rest is the tendency of a body to **stay at rest**.

e.g. A passenger stands in a train that is at rest. When the train **starts suddenly**, the passenger **falls backward**. The feet move forward with the train. But inertia keeps the **upper body at rest**.

Train starts with a jerk

Passenger falls backward

ii Inertia of Motion

Inertia of motion is the tendency of a moving body to **keep moving uniformly** along a straight line.

e.g. A passenger stands in a moving bus. The driver brakes hard and the bus **stops suddenly**. The passenger **falls forward**. The feet stop along with the bus. But inertia keeps the **upper body moving**.

Bus stops with a jerk

Passenger falls forward

iii Inertia of Direction

Inertia of direction is the inability of a body to **change its direction of motion by itself**.

e.g. An **umbrella** keeps us dry in the rain. The raindrops fall **vertically downward**. They cannot change their direction by themselves to wet us.

Raindrops falling vertically

Do not turn by themselves

1.6 Newton's First Law of Motion

Sir Isaac Newton (1642–1727) made a careful, step-by-step study of motion. He took Galileo's ideas further and stated **three laws of motion**. We call them **Newton's laws of motion**.

LAW

Newton's first law of motion states that a body at rest **remains at rest**, and a body in uniform motion **keeps moving uniformly along a straight line**, until an **external force** acts on it and changes this state.

NOTE

The First Law Defines Force Qualitatively

In the state of rest and in uniform motion, the acceleration is **zero**. So the first law can be written as:

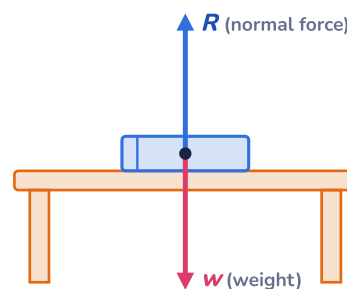
$$\vec{F}_{\text{net}} = 0 \Rightarrow \vec{a} = 0$$

So a body has **zero acceleration** whenever the net external force on it is zero. Its acceleration can be non-zero **only** when a **net external force** acts. In this way, the first law defines force **qualitatively**.

i Body at Rest

A body at rest **stays at rest**. It begins to move only when an **external force** acts on it.

e.g. A book lies at rest on a table. Two forces act on it. Its **weight** w acts downward due to gravity. The **normal force** R from the table acts upward. The book stays at rest, so the net force on it is zero. So $w = R$.

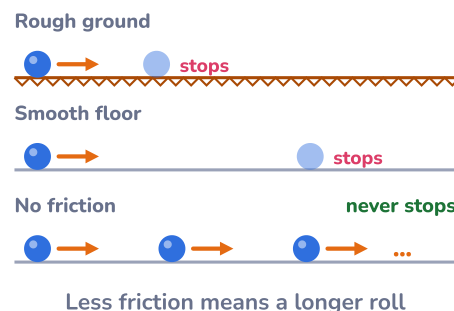


Book at rest on a table: $w = R$

ii Body in Motion

A moving body keeps its **uniform speed** and its **straight path** until an external force acts on it.

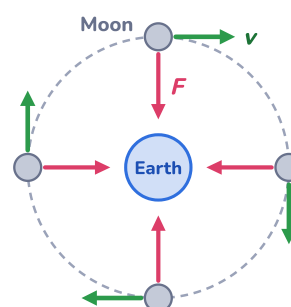
e.g. A rolling ball soon stops on **rough ground** because of **friction**. It rolls farther on a **smooth floor**. With **zero friction**, the ball would keep rolling **forever**.



iii Change of Direction

A body in uniform straight-line motion **cannot turn by itself**. Its direction changes only when a force acts **normal (perpendicular)** to its motion.

e.g. The direction of motion of the **Moon** changes all the time. The **gravitational pull of the Earth** on the Moon gives the force needed for this change.



Velocity v is along the orbit. The Earth's pull F points to the Earth

CONCEPT

The First Law Defines Inertia

By the first law, a body **by itself** can neither start moving, nor stop, nor turn. This property is inertia. So the first law defines **inertia**, and it is also called the **law of inertia**.

02

PART 2

Momentum, Second and Third Laws

2.1 Linear Momentum

DEFINITION

The **linear momentum** of a body is the **product of its mass m and its velocity $\vec{v}$** . It measures the **quantity of motion** in the body. It is written as $\vec{p}$.

MOMENTUM

$$\vec{p} = m\vec{v}$$

Nature

Vector quantity.

Unit

kg m s⁻¹ (SI); g cm s⁻¹ (CGS).

Dimensional Formula

[MLT⁻¹]

2.2 Variation of Momentum

Momentum changes with both the **mass** and the **velocity** of a body. The three cases below show how:

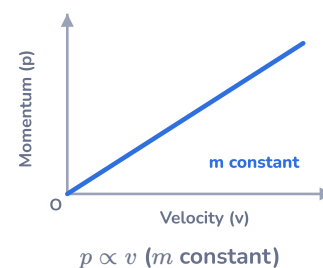
I Equal Masses ($v_1 > v_2$)

Two bodies have the **same mass m** but different velocities v_1 and v_2 . Then:

$$p_1 = mv_1, \quad p_2 = mv_2 \Rightarrow \frac{p_1}{p_2} = \frac{v_1}{v_2}$$

Equal masses

Momentum $\propto$ velocity

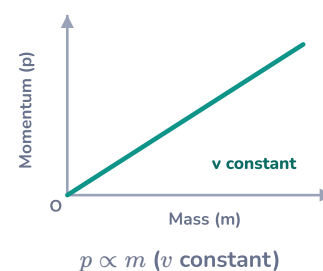
II Equal Velocities ($m_1 > m_2$)

A heavier body (m_1) and a lighter body (m_2) move with the **same velocity**, which is v for both. Then:

$$p_1 = m_1v, \quad p_2 = m_2v \Rightarrow \frac{p_1}{p_2} = \frac{m_1}{m_2}$$

Equal velocities

Momentum $\propto$ mass



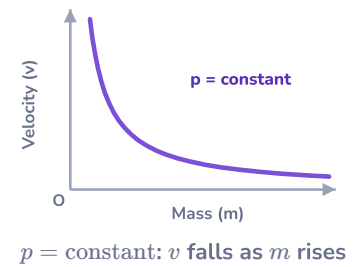
III Equal Momenta (p constant)

Two bodies have **equal linear momenta**:

$$p_1 = p_2 \Rightarrow m_1 v_1 = m_2 v_2 \Rightarrow \frac{v_2}{v_1} = \frac{m_1}{m_2}$$

Equal momenta

Velocity $\propto 1 / \text{mass}$



2.3 Momentum in Everyday Life

Many common observations depend on momentum:

i Truck and Car

To bring a **truck** and a car to equal speeds in equal times, the truck needs a **bigger force**. Stopping the truck from that speed also takes a bigger opposing force. The reason is that the truck is **heavier**.

ii Fast Bullet

A fast **bullet** from a gun easily goes through human tissue. At a **moderate speed**, the same bullet causes very little harm. So, for the same mass, a **higher speed** needs a bigger force to stop the body.

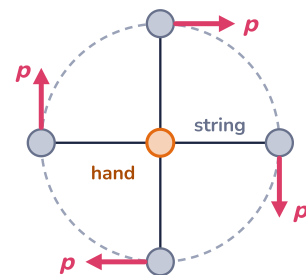
iii Light and Heavy Stones

Drop a light stone and a heavy stone from the top of a building. Catching the **light stone** is **easier**. So **mass** plays a key part in deciding how a force changes a body's motion.

iv Rotating Stone

A stone tied to a string is whirled in a **horizontal circle** at **uniform speed**. The **magnitude** of its momentum stays the **same**. But its **direction** keeps changing.

A force is needed to change this momentum vector. The hand gives this force through the string. A **greater force** is needed for a **greater speed** or a **smaller radius**. In both cases, the **rate of change of momentum** is greater.



The size of $\vec{p}$ stays the same, but its direction keeps changing

RESULT

All these examples point to one quantity, **momentum**, which is the **product of mass and velocity**. Momentum decides how much force is required to bring a body to rest.

2.4 Newton's Second Law of Motion

LAW

Newton's second law of motion states that the external force on a body is **directly proportional** to how fast its momentum changes, that is, to its **rate of change of momentum**. The momentum changes **in the direction of the applied force**.

SETUP – GIVEN INFORMATION

- ◆ A body of mass m is acted on by an external force $\vec{F}$ for a small time Δt .
- ◆ Its velocity changes from $\vec{v}$ to $\vec{v} + \Delta\vec{v}$.
- ◆ So the change in momentum is $\Delta\vec{p} = m \Delta\vec{v}$.

DERIVATION

- 1 By the second law, the force is proportional to the rate of change of momentum:

$$\vec{F} \propto \frac{\Delta\vec{p}}{\Delta t} \Rightarrow \vec{F} = k \frac{\Delta\vec{p}}{\Delta t}$$

Here k is the constant of proportionality.

- 2 Let $\Delta t \rightarrow 0$. The ratio becomes a derivative:

$$\vec{F} = k \frac{d\vec{p}}{dt}$$

- 3 For a body of constant mass m , and since $\frac{d\vec{v}}{dt} = \vec{a}$ (acceleration):

$$\vec{F} = k \frac{d(m\vec{v})}{dt} = k m \frac{d\vec{v}}{dt} = k m \vec{a}$$

- 4 Choose the units so that $k = 1$:

$$\vec{F} = m\vec{a}$$

NEWTON'S SECOND LAW

$$\vec{F} = m\vec{a}$$

$\therefore$ 1 unit force = 1 unit mass $\times$ 1 unit acceleration

NOTE

Unit Force

One **unit of force** gives a body of **unit mass** an acceleration of **one unit**. Force has both size and direction, so it is a **vector**. Its SI unit is the **newton (N)**.

• UNITS OF FORCE

KIND · SYSTEM	UNIT	MEANING
Absolute · SI	newton (N)	$1 \text{ N} = 1 \text{ kg} \times 1 \text{ m s}^{-2}$. It gives an acceleration of 1 m s^{-2} to a mass of 1 kg .
Absolute · CGS	dyne	$1 \text{ dyne} = 1 \text{ g} \times 1 \text{ cm s}^{-2}$. It gives an acceleration of 1 cm s^{-2} to a mass of 1 g .
Gravitational · SI	kg-wt (kgf)	$1 \text{ kg-wt} = 1 \text{ kgf} = 9.8 \text{ N}$. It gives an acceleration of 9.8 m s^{-2} to a mass of 1 kg .
Gravitational · CGS	g-wt (gf)	$1 \text{ g-wt} = 1 \text{ gf} = 980 \text{ dyne}$. It gives an acceleration of 980 cm s^{-2} to a mass of 1 g .

RELATION BETWEEN NEWTON AND DYNE

$$1 \text{ N} = 1000 \text{ g} \times 100 \text{ cm s}^{-2} = 10^5 \text{ g cm s}^{-2} = 10^5 \text{ dyne}$$

REMEMBER

One **gravitational unit** equals g times the **absolute unit** of the same system. Gravitational units are also known as **practical units**.

e.g. A body of mass 10 kg has a weight of 10 kg-wt .

• SECOND LAW IN COMPONENT FORM

The force, momentum and acceleration vectors can be written in terms of their **rectangular components**:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \quad \vec{p} = p_x \hat{i} + p_y \hat{j} + p_z \hat{k} \quad \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

From $\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$, equate the components along the three axes:

ALONG X

$$F_x = \frac{dp_x}{dt} = ma_x$$

ALONG Y

$$F_y = \frac{dp_y}{dt} = ma_y$$

ALONG Z

$$F_z = \frac{dp_z}{dt} = ma_z$$

NOTE

These three equations are the second law in **component form**. Along each axis, force = **rate of change of momentum** = mass $\times$ **acceleration**, all taken along that axis.

2.5 Impulse

DEFINITION

Impulse measures the effect of a **large force** that acts for a **very short time** and produces a **finite change in momentum**. It is equal to the **average force multiplied by the time** for which the force acts. Its symbol is $\vec{I}$.

IMPULSE

$$\vec{I} = \vec{F}_{\text{av}} \times t = \vec{F}_{\text{av}} \Delta t$$

Nature

Vector quantity. Its direction is the **same as the force**.

SI Unit

newton-second (N · s)

Dimensional Formula

[MLT⁻¹]

● IMPULSE-MOMENTUM THEOREM

SETUP – GIVEN INFORMATION

- ◆ Let a force $\vec{F}$ act on a body between the times t_1 and t_2 .
- ◆ $\vec{F}_{\text{av}}$ is the average force during this interval.

DERIVATION

- 1 Impulse in a very small time dt :

$$d\vec{I} = \vec{F} dt$$

- 2 Integrate over the whole interval:

$$\vec{I} = \int_{t_1}^{t_2} \vec{F} dt$$

- 3 For the average force:

$$\vec{I} = \vec{F}_{\text{av}} \int_{t_1}^{t_2} dt = \vec{F}_{\text{av}}(t_2 - t_1) = \vec{F}_{\text{av}} \Delta t$$

- 4 By Newton's second law:

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \vec{F} dt = d\vec{p}$$

- 5 Integrate both sides:

$$\int_{t_1}^{t_2} \vec{F} dt = \int_{p_1}^{p_2} d\vec{p} \Rightarrow \vec{I} = \vec{p}_2 - \vec{p}_1$$

IMPULSE-MOMENTUM THEOREM

$$\vec{I} = \vec{p}_2 - \vec{p}_1 = \Delta \vec{p}$$

The impulse given by a force equals the **total change in momentum** it causes.

2.6 Applications of Impulse

In daily life, the **time of contact** is often changed on purpose:

i Ball Bouncing off a Wall

The wall pushes the ball for a **very short time**, yet it **reverses the ball's momentum**. Force and time are hard to measure alone. Their **product** $F \times t$ (the change in momentum) can be measured.

Contact lasts a very short time

Large force

ii Athlete Stopping Slowly

After a fast race, an athlete should **slow down gradually**, not stop at once. Stopping then takes **more time**, so the force on the body is **smaller**.

Gradual stop (time $\uparrow$)

Smaller force

iii Buffers in Train Bogies

Railway bogies have **buffers**. During shunting, they make each impact last **longer**. A longer impact time means a **smaller force**, so there are no severe jerks.

Longer impact (buffers)

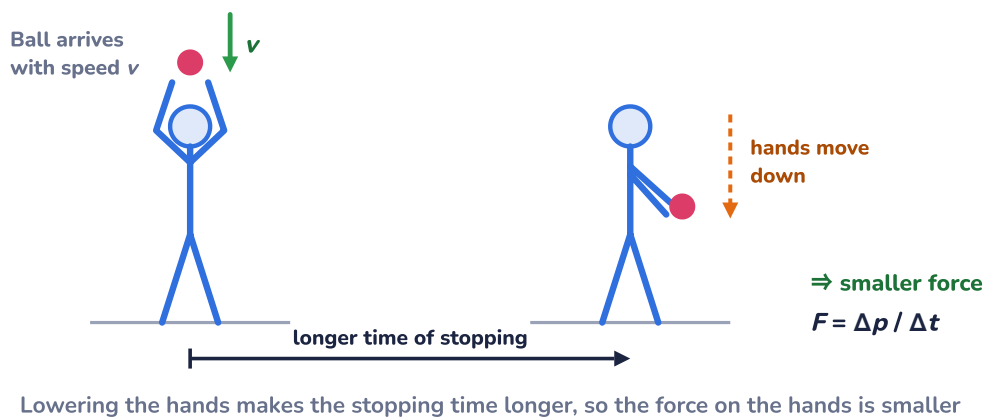
Smaller force

iv Cricketer Lowering Hands

A cricketer **moves the hands down** while catching a fast ball. The ball takes **more time** to stop, so a **smaller retarding force** gives the same change in momentum. The hands are **not hurt**.

Hands lowered

Less force, no injury



2.7 Newton's Third Law of Motion

Press a **coiled spring** with your hand. The force of your hand compresses the spring. The compressed spring also **pushes back** on your hand. In the same way, the Earth pulls a stone downward. The stone also pulls the Earth with an **equal and opposite force**. We do not notice it because the Earth is very massive. So forces always occur in **pairs**. They come from the **mutual interaction** of two bodies.

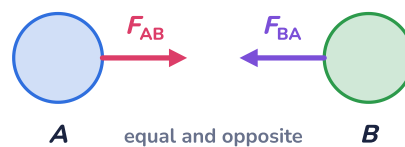
LAW

Newton's third law of motion states that when one body exerts a force on another (the **action**), the second body exerts a force on the first that is **equal in size and opposite in direction** (the **reaction**).

Forces in nature always come in pairs. Body B acts on body A with some force. Body A then acts on body B with a force of the **same size** in the **opposite direction**. In symbols:

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Here $\vec{F}_{AB}$ is the force on A by B , and $\vec{F}_{BA}$ is the force on B by A .



Action-reaction pair on bodies A and B

● IMPORTANT POINTS ABOUT THE THIRD LAW

i Any Kind of Force

The third law is true for **every type of force**. The action-reaction pair can be **mechanical**, **gravitational**, **electric** or any other kind.

ii Two Different Bodies

The action acts on one body and the reaction acts on the **other body**. They never act on the same body. If they did, they would add up to **zero**, and no body could ever accelerate.

iii They Never Cancel

Even though action and reaction are equal and opposite, they **cannot cancel**. Each one acts on a **different body**.

iv No Action Without Reaction

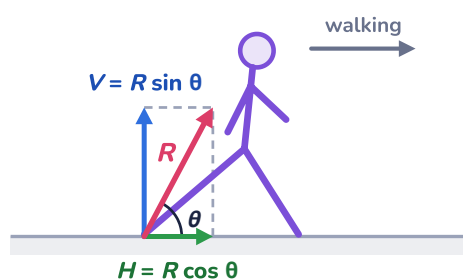
A force cannot act alone, so there is no action **without a reaction**. In a game of **tug of war**, one side can pull hard only because the other team holds and pulls the opposite end. If that end were left free, the team could not apply any force.

2.8 Examples of the Third Law

The third law explains many everyday events:

i Walking

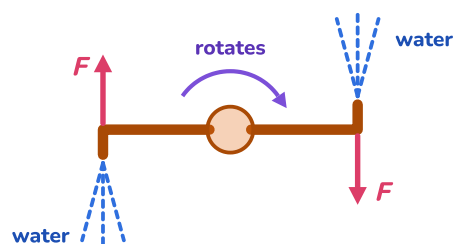
When we walk, each foot **pushes the ground backward**. This push is the **action**. The ground pushes back on the foot with an **equal and opposite reaction** R . Its **vertical component** $V = R \sin \theta$ supports our weight. Its **horizontal component** $H = R \cos \theta$ moves us **forward**.



Reaction R on the foot and its two components

ii Rotatory Lawn Sprinkler

A **rotatory lawn sprinkler** is a good example of the third law. Water rushes out through the bent **nozzles**. In return, the water pushes each nozzle **backward** with an **equal and opposite force**. These backward pushes **turn the sprinkler** the other way, and it sprays water all around.



Top view of a rotating sprinkler

2.9 Apparent Weight in a Lift

Think of a man of mass m standing on a **weighing machine** inside a lift. He presses on the machine with his **actual weight** mg . The machine pushes him back with a **reaction** R , and its reading shows this value of R .

DEFINITION

Apparent weight is the **reaction** R that the surface of contact exerts on a person. Let us see how R compares with mg in different cases.

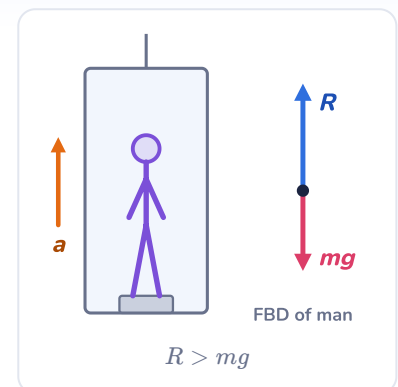
i Lift Accelerating Upward

The net force on the man acts **upward**:

$$R - mg = ma \Rightarrow R = m(g + a)$$

$$R = m(g + a)$$

Apparent weight increases $\uparrow$



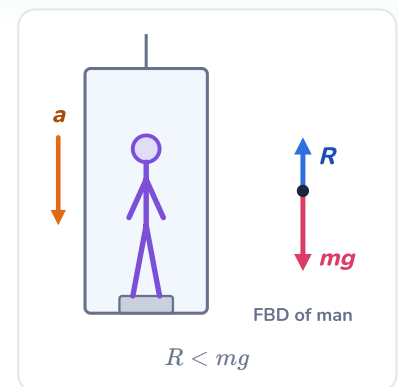
ii Lift Accelerating Downward

The net force on the man acts **downward**:

$$mg - R = ma \Rightarrow R = m(g - a)$$

$$R = m(g - a)$$

Apparent weight decreases $\downarrow$



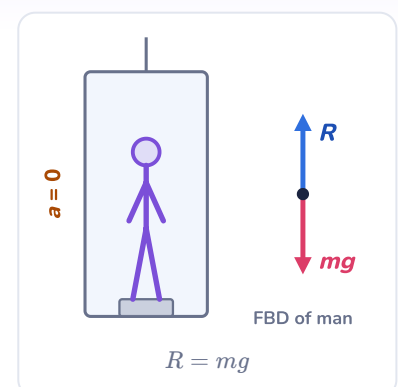
iii Lift at Rest or with Uniform Velocity

The acceleration is zero, $a = 0$. So:

$$R - mg = m \times 0 \Rightarrow R = mg$$

$$R = mg$$

Equal to the actual weight



iv Free Fall ($a = g$)

Suppose the cable holding the lift snaps. The lift then **falls freely**, so $a = g$. Then:

$$R = m(g - a) = m(g - g) = 0$$

$$R = 0$$

Weightlessness

2.10 Conservation of Momentum

This law follows from the **second** and **third** laws of Newton.

PRINCIPLE

The **law of conservation of momentum** states that the **total momentum** of a system **never changes with time** as long as no external force acts on it, that is, when $\vec{F}_{\text{ext}} = 0$.

NOTE

In an **isolated system**, the particles push and pull each other with **internal forces**. These forces change the momentum of each particle. But they come in **equal and opposite pairs** (Newton's third law). So they **cancel in pairs**, and the total momentum does not change.

• PROOF FOR TWO PARTICLES

Consider **two particles** that exert forces on each other. We will prove that their **total linear momentum** does not change when no external force acts. The result holds whatever the strength of the forces between them.

SETUP – GIVEN INFORMATION

- ◆ Particle 1 has mass m_1 and momentum $\vec{p}_1$. Particle 2 has mass m_2 and momentum $\vec{p}_2$.
- ◆ Forces $\vec{F}_1$ and $\vec{F}_2$ act on them. In a small time Δt , their momenta change by $\Delta\vec{p}_1$ and $\Delta\vec{p}_2$.
- ◆ Every force has two parts. The **external** part comes from bodies outside the system. The **internal** part is the force that the two particles exert on each other.

DERIVATION

- 1 The total momentum is the vector sum of the two momenta:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \quad \dots \text{(i)}$$

- 2 In the small time Δt , the total change in momentum is:

$$\Delta\vec{p} = \Delta\vec{p}_1 + \Delta\vec{p}_2$$

3 Divide by Δt and take the limit $\Delta t \rightarrow 0$:

$$\frac{d\vec{p}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt}$$

4 By the second law, each rate of change of momentum equals the force on that particle:

$$\frac{d\vec{p}}{dt} = \vec{F}_1 + \vec{F}_2 \quad \dots \text{(ii)}$$

5 Split each force into its external and internal parts:

$$\vec{F}_1 = \vec{F}_{1,\text{ext}} + \vec{F}_{1,\text{int}} \quad \text{and} \quad \vec{F}_2 = \vec{F}_{2,\text{ext}} + \vec{F}_{2,\text{int}}$$

6 Put these in (ii) and regroup:

$$\frac{d\vec{p}}{dt} = (\vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}}) + (\vec{F}_{1,\text{int}} + \vec{F}_{2,\text{int}}) \quad \dots \text{(iii)}$$

7 The two internal forces form an action–reaction pair, so by the third law they cancel:

$$\vec{F}_{1,\text{int}} = -\vec{F}_{2,\text{int}} \Rightarrow \vec{F}_{1,\text{int}} + \vec{F}_{2,\text{int}} = 0$$

8 So only the external forces remain:

$$\frac{d\vec{p}}{dt} = \vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}} = \vec{F}_{\text{ext}}$$

9 An **isolated system** has no external force on it, so $\vec{F}_{\text{ext}} = 0$. Therefore:

$$\frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \text{constant}$$

Hence the total momentum never changes with time. It stays conserved.

LAW OF CONSERVATION OF MOMENTUM

$$\vec{F}_{\text{ext}} = 0 \Rightarrow \vec{p} = \text{constant}$$

Whenever $\vec{F}_{\text{ext}} = 0$, the linear momentum of the system is conserved, whatever the internal forces may be.

• APPLICATIONS OF CONSERVATION OF MOMENTUM

i Recoil of a Gun

As the gun fires, it pushes the bullet forward with a force $\vec{F}$. By the third law, the bullet pushes the gun back with $-\vec{F}$ for the same time Δt . Gun and bullet were at rest at first, so their total momentum stays zero:

$$\vec{p}_b + \vec{p}_g = 0 \Rightarrow \vec{p}_g = -\vec{p}_b$$

So the gun **recoils (moves backward)**. Its momentum is equal and opposite to that of the bullet.

ii Explosion of a Bomb

Before it explodes, a bomb at rest has **zero momentum**. Suppose it breaks into **two equal pieces**. The pieces move apart in **exactly opposite directions** at **equal speeds**. Their momenta cancel, so the total momentum is still **zero**.

The explosion forces act inside the bomb, so they are **internal** forces. No force from outside acts on the bomb.

03 PART 3 Equilibrium, Friction and Circular Motion

3.1 Equilibrium of a Particle

DEFINITION

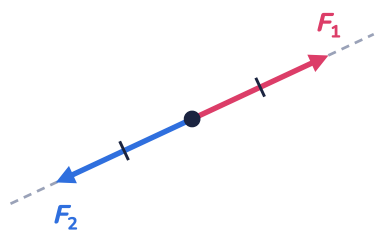
Concurrent forces are forces that act at a **single point** or on a single particle. Such forces are in **equilibrium** when their **resultant (vector sum)** is zero.

• CONDITIONS FOR EQUILIBRIUM

i Two Concurrent Forces

A pair of forces $\vec{F}_1$ and $\vec{F}_2$ balance each other only when they have the **same size** and **opposite directions**:

$$\vec{F}_1 = -\vec{F}_2$$

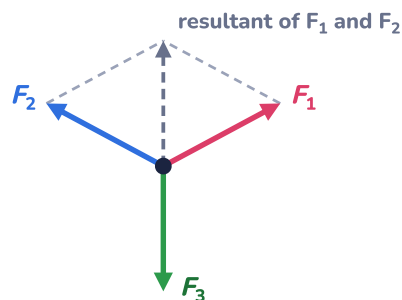


Equal (tick marks) and opposite forces

ii Three Concurrent Forces

For three forces, find the **resultant of any two** by the **parallelogram law**. The forces balance when this resultant is equal and opposite to the **third force**:

$$\vec{F}_1 + \vec{F}_2 = -\vec{F}_3 \Rightarrow \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$



The resultant of $\vec{F}_1$ and $\vec{F}_2$ balances $\vec{F}_3$

iii n Concurrent Forces

Many forces (n of them) keep a particle in equilibrium when, drawn head to tail in order, they form a **closed polygon**:

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n = 0$$

So the sum of the components along **each axis** is zero:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$

Here, for example, $\sum F_x = F_{1x} + F_{2x} + \cdots + F_{nx}$. F_{1x}, F_{1y}, F_{1z} are the rectangular components of $\vec{F}_1$ along the x, y and z directions.

3.2 Lami's Theorem

LAW

Lami's theorem states that when **three concurrent forces** $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ keep a body in **equilibrium**, each force divided by the **sine of the angle between the other two forces** gives the same value.

LAMI'S THEOREM

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$$

α

Angle between $\vec{F}_2$ and $\vec{F}_3$

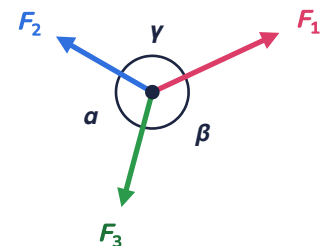
β

Angle between $\vec{F}_3$ and $\vec{F}_1$

γ

Angle between $\vec{F}_1$ and $\vec{F}_2$

Each force sits opposite to its own angle. For example, α lies between the other two forces, $\vec{F}_2$ and $\vec{F}_3$.

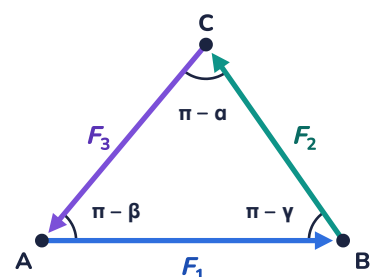


Three concurrent forces at angles α, β, γ

PROOF

SETUP – GIVEN INFORMATION

- The forces $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ balance one another. So, drawn head to tail in the same order, they form the **sides of a triangle ABC**.
- The interior angles of the triangle are $\pi - \alpha, \pi - \beta$ and $\pi - \gamma$.



Forces as the sides of $\triangle ABC$

DERIVATION

1 Apply the **law of sines** to $\triangle ABC$:

$$\frac{F_1}{\sin(\pi - \alpha)} = \frac{F_2}{\sin(\pi - \beta)} = \frac{F_3}{\sin(\pi - \gamma)}$$

2 Since $\sin(\pi - \theta) = \sin \theta$:

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$$

This is Lami's theorem.

3.3 Common Forces in Mechanics

The many forces in mechanics are of **two broad types**: **non-contact** forces and **contact** forces.

Non-Contact Force

Gravity pulls every object on the Earth. In mechanics, the **gravitational force** is the only force that acts **without contact**.

Contact Forces

All other common forces come from **touching** a solid or a fluid. Such contact forces between two bodies obey **Newton's third law**.

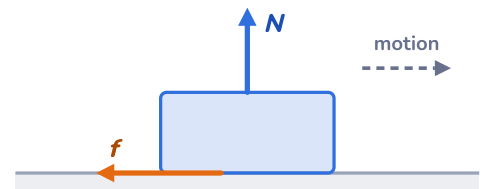
• PARTS OF A CONTACT FORCE

Normal Reaction N

The component of the contact force at **right angles (normal)** to the touching surfaces.

Friction f

The component of the contact force **along (parallel to)** the touching surfaces.



A block with normal reaction N and friction f

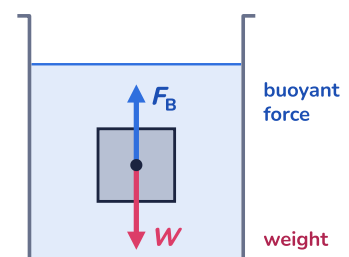
• FORCES DUE TO FLUIDS

Buoyant Force

A fluid pushes **upward** on a solid placed in it. This push equals the **weight of the fluid displaced**.

Viscous Force and Air Resistance

The **viscous force** opposes motion through a fluid. The viscous force and **air resistance** are also contact forces.



Upward buoyant force on an immersed body

3.4 Tension and Spring Force

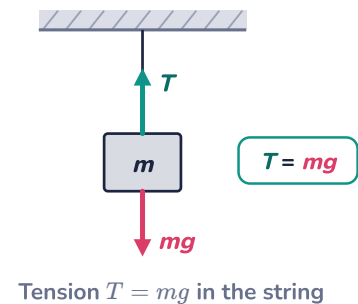
Strings and springs also exert **contact forces**. These are the **tension** in a string and the **spring force**:

Tension in a String

Hang a body of mass m from a string. The string pulls it **upward** with a force called **tension** T , which balances its weight:

$$T = mg$$

Here g is the acceleration due to gravity and T is the tension in the string.

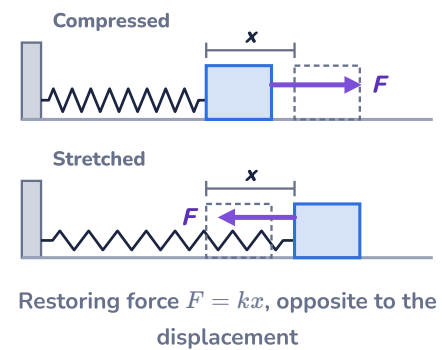


Spring Force

When an external force squeezes or stretches a spring, the spring pushes or pulls back with a **restoring force**. For small displacements, this force is **proportional to the displacement**:

$$F = kx$$

Here k is the **spring constant** and x is the displacement. The restoring force acts opposite to the displacement.



NOTE

- ◆ Among the forces of mechanics, only **gravity** acts **without contact**.
- ◆ For a **heavy (not light)** rope or string, include its **weight** as well.

3.5 Friction

DEFINITION

Friction is the force that comes into play **along the surface of contact** when one body slides, or tends to slide, over another. It **opposes their relative motion**.

Push a wooden block **gently** along a horizontal table. It moves a little and then **comes to rest**. Its velocity has changed. Newton's second law tells us that some **retarding force** acts on it. This force is **friction**.

Friction acts **along (tangential to)** the touching surfaces. It always points **against the relative motion**.



Friction points against the motion

● TYPES OF FRICTION

1 Internal Friction

It acts between **two layers of a liquid** that move relative to each other. Its other name is the **viscosity** of the liquid.

2 External (Contact) Friction

It acts between two **touching bodies** that move, or are about to move, over each other. Its **three kinds** are **static**, **limiting** and **kinetic** friction.

● (I) STATIC FRICTION

Push a block lying on a table with a **small force F** . The block **stays where it is**, because a friction force f_s appears and **balances** the push.

Static friction (f_s) is the friction on a body that has **not yet started** to slide over another body. It opposes the **impending motion**, that is, the motion that would start if there were no friction.

Static friction **adjusts itself** to match the push, so it is a **self-adjusting force**. If nothing pushes the block, static friction is **zero**.



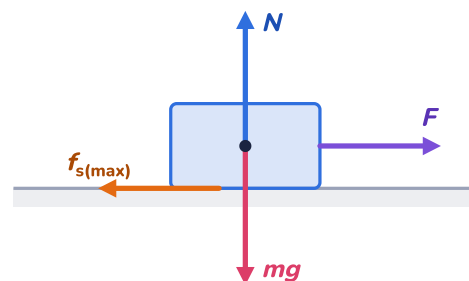
Static friction f_s balances the applied force F

● (II) LIMITING FRICTION

Slowly increase the push. Static friction f_s **grows** with it, and the block stays at rest.

At one particular push, the block is **just about to slide**. Static friction has now reached its **largest value**. **Limiting friction $f_{s(\max)}$** is this **maximum value of static friction**.

$$f_s \leq f_{s(\max)}$$



Limiting friction on a body about to move

3.6 Laws of Limiting Friction

a Nature of Surfaces

Limiting friction is decided by the **nature (kind) of the surfaces** in contact and by **how rough** they are.

b Direction

It acts **along the contact surfaces (tangential)**, **against** the direction in which the body is about to move.

c Independent of Area

Changing the **area of contact** does not change it, provided the **normal reaction** is unchanged.

d Proportional to Normal Reaction

It increases in **direct proportion** to the normal reaction R . Hence:

$$f_{s(\max)} \propto R \Rightarrow f_{s(\max)} = \mu_s R$$

• COEFFICIENT OF STATIC FRICTION

DEFINITION

The **coefficient of static friction** (μ_s) of two surfaces is their **limiting friction divided by the normal reaction**. It is the constant in law (d).

COEFFICIENT OF STATIC FRICTION

$$\mu_s = \frac{f_{s(\max)}}{R} = \frac{\text{Limiting friction}}{\text{Normal reaction}}$$

For a block on a horizontal surface, $R = w = mg$ (m = mass of the block, g = acceleration due to gravity).

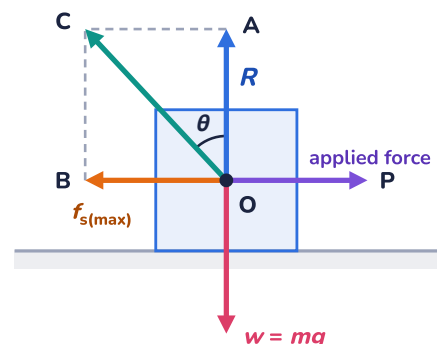
3.7 Angle of Friction

DEFINITION

The angle between the **normal reaction** and the **resultant** of the normal reaction and the limiting friction is called the **angle of friction** (θ).

SETUP – GIVEN INFORMATION

- ◆ OA = normal reaction R , OB = limiting friction $f_{s(\max)}$ and OC = their resultant.
- ◆ θ is the angle between the normal reaction R and the resultant OC .



OC is the resultant of R and $f_{s(\max)}$. It makes angle θ with R

DERIVATION

1 In $\triangle AOC$:

$$\tan \theta = \frac{AC}{OA} = \frac{OB}{OA} = \frac{f_{s(\max)}}{R}$$

2 But $\frac{f_{s(\max)}}{R} = \mu_s$. So:

$$\tan \theta = \mu_s \Rightarrow \theta = \tan^{-1}(\mu_s)$$

ANGLE OF FRICTION

$$\tan \theta = \mu_s \Rightarrow \theta = \tan^{-1}(\mu_s)$$

So the tangent of the angle of friction gives the **coefficient of static friction**.

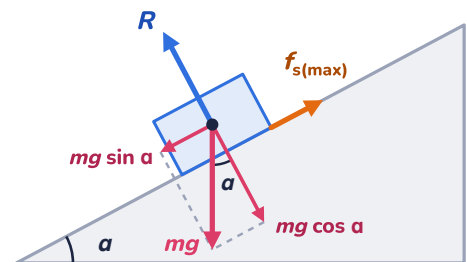
3.8 Angle of Repose

DEFINITION

The **angle of repose** (α) is the **smallest angle** with the horizontal to which an inclined plane must be tilted so that a body resting on it **just starts sliding down**. It depends on the **materials and nature** of the two surfaces.

SETUP – FORCES ON THE BODY

- Put a body of mass m on an inclined plane. Tilt the plane until the body is just about to slide. That angle is α .
- Its weight mg acts vertically downward.



Body on an incline at the angle of repose α

DERIVATION

1 Limiting friction acts **up the incline**. It balances the part of the weight along the incline:

$$f_{s(\max)} = mg \sin \alpha \quad \dots (i)$$

2 The normal reaction balances the part of the weight perpendicular to the incline:

$$R = mg \cos \alpha \quad \dots (ii)$$

3 Divide (i) by (ii):

$$\frac{f_{s(\max)}}{R} = \frac{mg \sin \alpha}{mg \cos \alpha} \Rightarrow \mu_s = \tan \alpha$$

ANGLE OF REPOSE

$$\mu_s = \tan \alpha \quad \text{and} \quad \theta = \alpha$$

The tangent of the angle of repose gives the **coefficient of static friction**. It also shows that the angle of friction and the angle of repose are **equal**.

3.9 Kinetic Friction

DEFINITION

Kinetic friction (also called **dynamic friction**) is the friction between two surfaces that are **already sliding** over each other. It acts against their **relative motion**.

It takes over once the applied force becomes slightly more than the limiting friction. Its value is **slightly smaller** than the limiting friction.

LAWS OF KINETIC FRICTION

a Constant Value

It acts against the relative motion. For a given pair of surfaces, its value is **fixed (constant)**. This value depends on the **nature of the surfaces**.

b Independent of Area

A bigger or smaller **area of contact** makes no difference, if the **normal reaction** is the same.

c Independent of Velocity

Its value is the same at **any speed** of sliding. It does not depend on velocity.

d Proportional to Normal Reaction

It grows in **direct proportion** to the normal reaction R . Hence:

$$f_k \propto R \Rightarrow f_k = \mu_k R$$

COEFFICIENT OF KINETIC FRICTION

$$\mu_k = \frac{f_k}{R} = \frac{\text{Kinetic friction}}{\text{Normal reaction}}$$

Divide the kinetic friction by the normal reaction to get μ_k .

REMEMBER

Kinetic friction is smaller than the limiting friction, so $f_k < f_{s(\max)}$. Therefore $\mu_k < \mu_s$. For the same two surfaces, μ_k is always **smaller** than μ_s .

3.10 Types of Kinetic Friction

There are **two kinds** of kinetic friction: sliding and rolling.

i Sliding Friction

Sliding friction is the friction on a body that **slides** along another surface.

e.g. When a flat block is dragged across a flat table top, sliding friction opposes it.

ii Rolling Friction

Rolling friction is the friction on a body that **rolls** along another surface.

e.g. Rolling wheels, circular discs, rings and spheres all feel rolling friction.

• LAWS OF SLIDING FRICTION

- ◆ It acts **against the applied force**. Its value is **constant** and is fixed by the kind of surfaces that slide on each other.
- ◆ It grows in **direct proportion** to the normal reaction R .
- ◆ The **area of contact** does not affect it, provided the normal reaction stays unchanged.
- ◆ Its value is **not affected by velocity**.

• WHY ROLLING FRICTION IS SMALL

CONCEPT

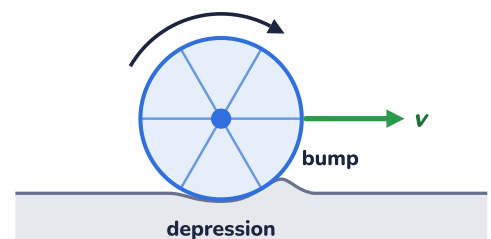
When a wheel rolls **without slipping**, its rim and the road **never rub** against each other. At each instant, the point of the wheel that touches the road is **at rest** relative to the road. So, ideally, no sliding (static) friction acts on the wheel.

Only **rolling friction** remains to be overcome, and it is **far smaller** than sliding friction. The **wheel** replaces sliding with rolling, so it cuts down friction greatly. For this reason, the wheel is counted among the **greatest inventions**.

• CAUSE OF ROLLING FRICTION

A rolling wheel rests on a **tiny area** of the road, so the **pressure** there is large. The road gets pressed into a **depression** under the wheel, and a small **bump** rises just ahead of it.

The depression, the bump and the **adhesive force** between the surfaces together hold the wheel back. This **retarding force** is **rolling friction**.



Depression and bump slow the wheel

3.11 Methods of Reducing Friction

Friction can be reduced in these ways:

i Lubrication

Lubricants like **oil** and **grease** fill the tiny hollows (**irregularities**) of the surfaces and make them smoother. This **lowers** the kinetic friction in a machine.

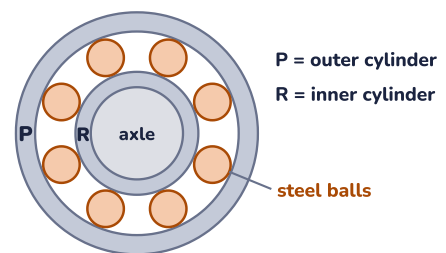
ii Polishing

A **polished** surface is **smoother**, so there is less friction.

iii Ball Bearing

In a **ball bearing**, hard **steel balls** sit between two **co-axial cylinders**, P and R .

As the wheel turns, the balls **roll**. The wheel no longer **slides on the axle**. It rides on the rolling balls instead. **Rolling friction** takes the place of sliding friction, so less power is wasted. The **power dissipation** falls.



Parts of a ball bearing

iv Air Cushion

A **thin layer (cushion) of air** is kept between two solid surfaces that move over each other. The air stops them from touching and **rubbing**.

v Streamlining

Streamlining the shape of a body that moves through air greatly reduces the friction due to **air**.

3.12 Dynamics of Circular Motion

A body moves in a circle of radius r with uniform speed v . Its acceleration is $\frac{v^2}{r}$, directed **towards the centre**. By Newton's second law, a force must produce this acceleration. This force is the **centripetal force**.

DEFINITION

Centripetal force is the force that keeps a body of mass m moving on a circular path. It always points **towards the centre** of the circle. The quantity $\frac{v^2}{r}$ is called the **centripetal (radial) acceleration**.

CENTRIPETAL FORCE

$$F_c = \frac{mv^2}{r} = mr\omega^2$$

since $v = \omega r$, where m = mass, v = speed, r = radius, ω = angular velocity

• SOURCES OF CENTRIPETAL FORCE

Tension

When a **stone on a string** is whirled in a circle, the **tension** in the string supplies the centripetal force.

Gravitation

A **planet** goes round the Sun because the Sun's **gravitational pull** supplies the centripetal force.

Friction

When a **car turns** on a flat (level) road, **friction** between its tyres and the road supplies the centripetal force.

3.13 Car on a Level Circular Road

A car of weight mg moves with speed v round a level curve of radius r . It tends to **skid outward**. **Friction** from the road on the tyres supplies the centripetal force that it needs.

(i) Weight mg

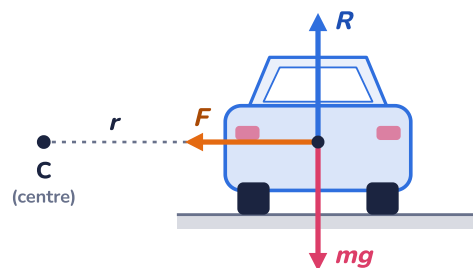
Vertically downward

(ii) Normal reaction R

Vertically upward

(iii) Friction F

Towards the centre



Car on a level circular road (rear view)

SETUP – GIVEN INFORMATION

- ◆ A car of mass m moves with speed v on a level circular road of radius r .
- ◆ The car does not accelerate up or down (in the vertical direction).

DERIVATION

1 The vertical forces balance:

$$R = mg$$

2 Static friction supplies the centripetal force. It can be at most the limiting friction F of the road. So:

$$\frac{mv^2}{r} \leq F$$

3 The limiting (maximum) friction is:

$$F = \mu_s R = \mu_s mg$$

4 Hence:

$$\frac{mv^2}{r} \leq \mu_s mg \Rightarrow v \leq \sqrt{\mu_s r g}$$

MAXIMUM SAFE SPEED ON A LEVEL ROAD

$$v_{\max} = \sqrt{\mu_s r g}$$

This is the highest speed at which a vehicle can go round a level curve **without skidding** (μ_s = coefficient of static friction).

3.14 Car on a Banked Road

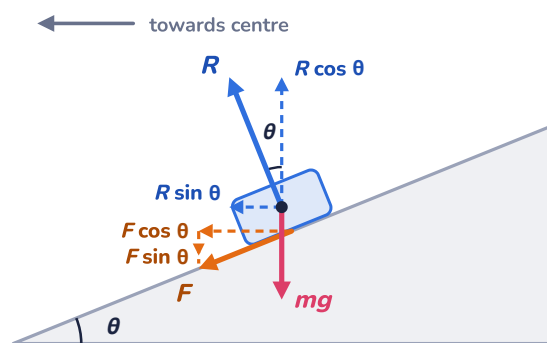
DEFINITION

Banking of a road is the raising of the **outer edge** of a curved road above its **inner edge**. The angle through which the road is raised is called the **angle of banking** (θ).

Banking reduces the **wear and tear of the tyres**. The road slopes up towards its outer edge. Then the **horizontal component of the normal reaction** gives the needed centripetal force.

Forces on the car: weight mg (downward), normal reaction R (perpendicular to the road) and friction F . Friction points **down the slope**, since the car tends to slide outward. Their components are:

- ◆ $R \cos \theta$ (upward) and $R \sin \theta$ (horizontal, towards the centre)
- ◆ $F \cos \theta$ (horizontal, towards the centre) and $F \sin \theta$ (downward)



Car on a road banked at angle θ

SETUP – GIVEN INFORMATION

- ◆ A car of mass m moves with speed v on a curved road of radius r , banked at angle θ .
- ◆ The car tends to slide **outward**, so friction on it points down the slope. At the top speed, friction is limiting: $F = \mu_s R$.

DERIVATION

1 The car does not move **vertically**. So the upward and downward components are equal:

$$R \cos \theta = mg + F \sin \theta \quad \dots (i)$$

- 2 The **horizontal components** together give the centripetal force:

$$R \sin \theta + F \cos \theta = \frac{mv^2}{r} \quad \dots \text{(ii)}$$

- 3 Put $F = \mu_s R$ in (i):

$$R \cos \theta = mg + \mu_s R \sin \theta \Rightarrow R(\cos \theta - \mu_s \sin \theta) = mg$$

$$\therefore R = \frac{mg}{\cos \theta - \mu_s \sin \theta} \quad \dots \text{(iii)}$$

- 4 Put $F = \mu_s R$ in (ii):

$$R \sin \theta + \mu_s R \cos \theta = \frac{mv^2}{r} \Rightarrow R(\sin \theta + \mu_s \cos \theta) = \frac{mv^2}{r} \quad \dots \text{(iv)}$$

- 5 Put R from (iii) into (iv):

$$\frac{mg(\sin \theta + \mu_s \cos \theta)}{\cos \theta - \mu_s \sin \theta} = \frac{mv^2}{r}$$

- 6 Solve for v^2 . The mass m cancels:

$$v^2 = \frac{rg(\sin \theta + \mu_s \cos \theta)}{\cos \theta - \mu_s \sin \theta}$$

- 7 Divide the numerator and the denominator by $\cos \theta$:

$$v^2 = \frac{rg(\tan \theta + \mu_s)}{1 - \mu_s \tan \theta}$$

This is the maximum safe speed. If the car goes faster than this, it skids outward (up the slope).

MAXIMUM SPEED ON A BANKED ROAD

$$v_{\max} = \sqrt{\frac{rg(\mu_s + \tan \theta)}{1 - \mu_s \tan \theta}}$$

• SPECIAL CASE – FRICTIONLESS BANKED ROAD

Optimum Speed ($\mu_s = 0$)

With no friction ($\mu_s = 0$), the car does not slide at this speed, even on a smooth road:

$$v = \sqrt{rg \tan \theta}$$

Angle of Banking

For the least wear and tear of the tyres, the angle of banking should be:

$$\theta = \tan^{-1} \left(\frac{v^2}{rg} \right)$$

NOTE

Suppose the road is smooth. If the speed is **less than** $\sqrt{rg \tan \theta}$, the car tends to slide **down** the slope, and the radius decreases. If the speed is **more**, the car tends to slide **up** the slope.

3.15 Solving Problems in Mechanics

Most mechanics problems involve an **assembly of bodies**, not a single body. The bodies push and pull on one another, and **gravity** acts on each of them.

DEFINITION

A **free body diagram (FBD)** is a drawing of **one chosen system on its own**, with **every force** on it marked. These forces come from the rest of the assembly and from any other outside agencies.

PROBLEM SOLVING STRATEGY

Steps to Solve a Mechanics Problem

1 Draw the assembly.

Make a rough sketch of the whole assembly. Show its **parts**, links, supports, etc.

2 Choose a system.

Pick one **convenient part** of the assembly and treat it as the system.

3 Draw the free body diagram.

Draw the chosen system by itself. Mark **every force** on it, from the rest of the assembly and from outside agencies. This drawing is the **free body diagram**.

4 Mark the known forces.

On the FBD, show the **magnitude and direction** of each known (given) force. The remaining forces are **unknowns**. Use the laws of motion to find them.

5 Repeat if needed.

Do the same for **other parts** of the system if required. Solving the equations of all the parts together gives the **required results**.