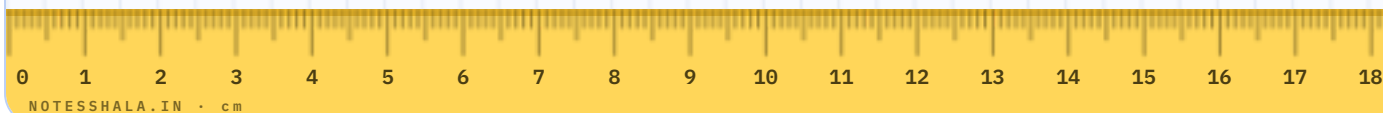


CHAPTER

03

Motion in a Plane

Scalars and Vectors • Motion in a Plane • Projectile and Circular Motion



What's inside

1 Scalars and Vectors	2	2 Motion in a Plane	21
1.1 Scalar and Vector Quantities	2	2.1 Position, Displacement and Velocity Vectors	21
1.2 Important Definitions Related to Vectors	4	2.2 Instantaneous Velocity	23
1.3 Position and Displacement Vectors	6	2.3 Acceleration in a Plane	23
1.4 Multiplication of a Vector by a Real Number	8	2.4 Instantaneous Acceleration	24
1.5 Resultant Vector	9	2.5 Motion in a Plane with Uniform Velocity	25
1.6 Addition of Vectors — Graphical Method	10	2.6 Motion in a Plane with Constant Acceleration	26
1.7 Subtraction of Vectors — Graphical Method	12	3 Projectile and Circular Motion	28
1.8 Resolution of a Vector in a Plane	13	3.1 Projectile Motion	28
1.9 Resolution of a Vector in Space	14	3.2 Equation of Path of a Projectile	29
1.10 Addition of Vectors — Analytical Method	16	3.3 Time of Flight	30
1.11 Subtraction of Vectors — Analytical Method	17	3.4 Maximum Height	30
1.12 Scalar Product (Dot Product)	18	3.5 Horizontal Range	31
1.13 Vector Product (Cross Product)	19	3.6 Projectile Thrown at an Angle to the Vertical	32
		3.7 Uniform Circular Motion	32
		3.8 Terms Related to Circular Motion	33
		3.9 Relation between Linear and Angular Velocity	34
		3.10 Centripetal Acceleration	35

Chapter 2 dealt with motion along a **straight line**. This chapter goes further, to motion in a **plane** (2-D) and motion in **space** (3-D). Such motion needs **vectors**, so we learn them first. After that comes motion with **constant acceleration** in a plane, mainly **projectile motion**. The last topic is **uniform circular motion**.

1.1 Scalar and Vector Quantities

Many quantities help us describe the physical world. Some of them are **mass, distance, displacement, speed, velocity, acceleration, momentum, force, energy, work** and **power**. Motion in **two** or **three dimensions** cannot be described without **vectors**. So we start with them.

Every physical quantity has a **magnitude** (size). Some quantities also have a **direction**. This gives two groups:

i Scalars

Scalars have only a **size (magnitude)**. They have no direction.

Magnitude

✓ Yes

Direction

✗ No

ii Vectors

Vectors have a **size (magnitude)** and also a **direction**.

Magnitude

✓ Yes

Direction

✓ Yes

• SCALAR QUANTITIES

DEFINITION

A physical quantity that has **only magnitude** and **no direction** is called a **scalar quantity**. A scalar is fully described by a **single number** with its **proper unit**.

Scalars follow the **ordinary rules of algebra**:

i Algebra of Scalars

Scalars are **added, subtracted, multiplied and divided** just like ordinary numbers.

ii Condition of Units

Only quantities with the **same unit** can be added or subtracted. Quantities with **different units** can be multiplied or divided.

• VECTOR QUANTITIES

DEFINITION

A physical quantity that has **both magnitude and direction** is called a **vector quantity**. Vectors are added and subtracted by the **triangle law** and the **parallelogram law**.

e.g. Force, velocity, displacement, acceleration, momentum, etc.

A vector is printed in **bold type**, like **$\mathbf{F}$** , **$\mathbf{a}$** , **$\mathbf{b}$** . It is also written with an **arrow over the letter**, like $\vec{F}$, $\vec{a}$, $\vec{b}$. We draw a vector as a straight line with an arrowhead:

How **long** the arrow is

Shows the magnitude

Which way the **arrowhead** points

Shows the direction

The magnitude is also called the **absolute value**

$$|\vec{v}| = v$$

EXAMPLE

Representing a Vector

A body moves with a velocity of **60 m/s due East**. Take a scale of **1 cm = 10 m/s**. Draw a line OP , **6 cm** long, towards East. Put an arrowhead at P . This line fully represents the velocity.

Point P is the **head** (terminal point). Point O is the **tail** (initial point) of $\vec{OP}$.



Vector OP is 6 cm long towards East: tail O , head P

Vectors are of **two types**:

1 Polar Vectors

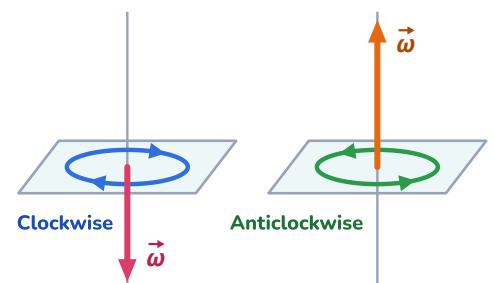
A vector that acts from a **starting point**, or at a **point of application**, is called a **polar vector**.

e.g. Displacement, force.

2 Axial Vectors

A vector that describes a **turning (rotational) effect** is called an **axial vector**. It lies **along the axis of rotation**. Which way it points on the axis depends on the turning: **clockwise** or **anticlockwise**.

e.g. Torque, angular velocity.



An axial vector lies along the axis of rotation (right-hand rule)

Tensors

Some quantities do not have **one fixed direction**. Yet their value **changes from one direction to another**. They are known as **tensors**.

e.g. Stress, pressure, surface tension, moment of inertia, etc.

1.2 Important Definitions Related to Vectors

These terms are used again and again with vectors.

i Modulus of a Vector

The **modulus** of a vector is simply its **magnitude**.

How we write it for $\vec{A}$

$|\vec{A}|$ or A

ii Unit Vector

A **unit vector** is a vector of magnitude **one (unity)** that points in a **definite direction**. The unit vector along $\vec{A}$ is $\hat{A}$, read as "**A cap**".

So every vector = its **magnitude** $\times$ the **unit vector** in its own direction.

UNIT VECTOR

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \Rightarrow \vec{A} = |\vec{A}| \hat{A}$$

Unit vector = $\frac{\text{vector}}{\text{magnitude of the vector}}$

Cartesian Unit Vectors

$\hat{i}$, $\hat{j}$ and $\hat{k}$ point along the **x, y and z axes**. Each has magnitude one.

Magnitude of a Unit Vector

It is equal to **one**. A unit vector has **no unit** and **no dimensions**.

iii Null Vector (Zero Vector)

A **null vector** (or **zero vector**) is a vector of **zero magnitude**. Its direction can be anything (**arbitrary**). It is written as $\vec{0}$.

Velocity of a body **at rest**

Null vector

Acceleration of a body moving at **constant velocity**

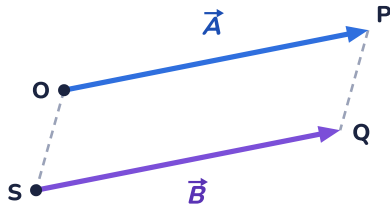
Null vector

iv Equal Vectors

Equal vectors have the **same magnitude** and point in the **same direction**.

$$\text{If } \overrightarrow{OP} = \overrightarrow{SQ}$$

$$\vec{A} = \vec{B}$$



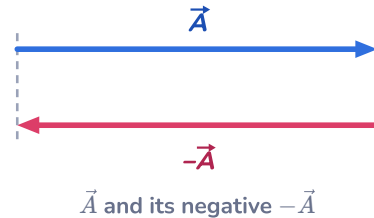
Equal vectors $\vec{A}$ and $\vec{B}$

v Negative Vector

A vector with the **same magnitude** as $\vec{A}$ but the **opposite direction** is called the **negative of $\vec{A}$** .

Negative of $\vec{A}$

$$-\vec{A}$$



NOTE

Free Vectors

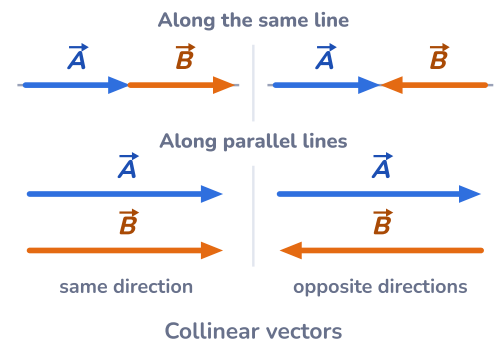
A vector is not tied to one place. Moving it **parallel to itself** leaves it unchanged. We call such vectors **free vectors**.

e.g. The velocity of a particle in straight-line motion.

vi Collinear Vectors

Collinear vectors lie on **one line** or on **parallel lines**. They can point the same way or opposite ways.

e.g. The pulls in a **tug of war**.

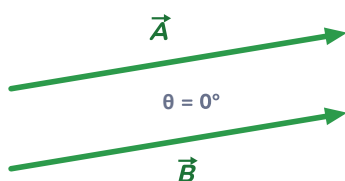


Parallel Vectors

Collinear vectors that point the **same way** are called **parallel vectors**.

Angle between the two

$$\theta = 0^\circ$$



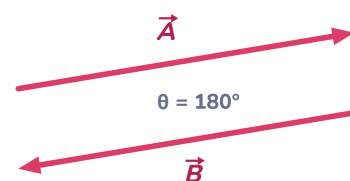
Parallel vectors

Anti-parallel Vectors

Collinear vectors that point **opposite ways** are called **anti-parallel vectors**.

Angle between the two

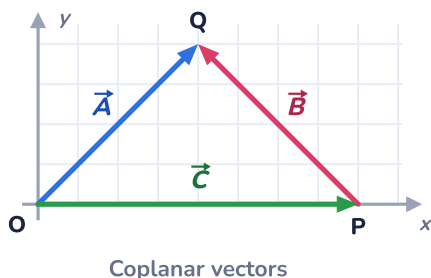
$$\theta = 180^\circ$$



Anti-parallel vectors

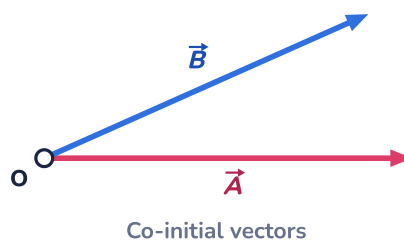
vii Coplanar Vectors

Vectors lying in **one plane** are called **coplanar vectors**. In the figure, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are all in the plane of the paper.



viii Co-initial Vectors

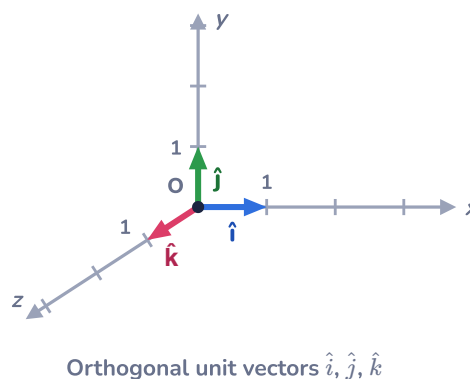
Vectors that start from **one common point** are called **co-initial vectors**. In the figure, $\vec{A}$ and $\vec{B}$ both begin at O.



ix Orthogonal Unit Vectors

Two or three unit vectors **at right angles** to one another are called **orthogonal unit vectors**. A common example is $\hat{i}$, $\hat{j}$ and $\hat{k}$, which lie along the x, y and z axes.

$$\hat{i} = \frac{\vec{x}}{x}, \quad \hat{j} = \frac{\vec{y}}{y}, \quad \hat{k} = \frac{\vec{z}}{z}$$



x Localised Vectors

A vector that must start from a **fixed point** is called a **localised vector**.

e.g. Position vector of a particle. It always starts at the origin.

xi Non-localised Vectors

A vector whose starting point is **free to change** is called a **non-localised vector**.

e.g. Velocity of a particle in straight-line motion.

1.3 Position and Displacement Vectors

• POSITION VECTOR

DEFINITION

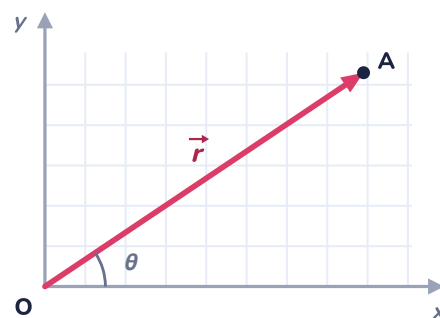
The **position vector** of an object is the vector drawn from the **origin** of the coordinate system to the object. Its symbol is $\vec{r}$.

Let an object move in the **xy-plane**, with the origin at O . At an instant t , the object is at point A . Its position vector at A is:

$$\overrightarrow{OA} = \vec{r}$$

A position vector gives **two pieces of information**:

- ◆ the **minimum distance** of the object from the origin O , and
- ◆ the **direction** of the object from the origin.



Position vector $\vec{r}$ of point A

DISPLACEMENT VECTOR

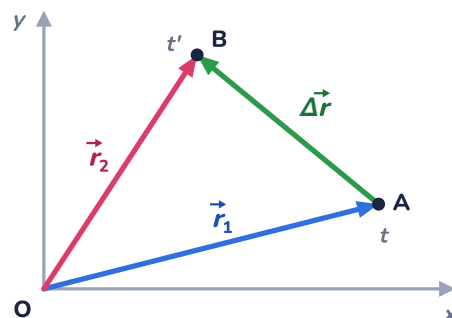
DEFINITION

The vector that shows **how much** and **in which direction** the position of an object changes in a time interval is called the **displacement vector**.

It is the **straight line** from the initial position to the final position. The path actually followed in between **does not matter**.

Let an object move in the **xy-plane**. At an instant t , it is at point A . At a later instant t' , it is at point B .

Its displacement from t to t' is $\overrightarrow{AB}$. Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) .



Displacement vector $\Delta\vec{r}$ from A to B

DERIVATION

1 The position vectors of the object at A and B are:

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} \quad \text{and} \quad \vec{r}_2 = x_2\hat{i} + y_2\hat{j}$$

2 The displacement $\overrightarrow{AB}$ is the change in the position vector:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

3 Put in the two position vectors:

$$\Delta\vec{r} = (x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})$$

DISPLACEMENT VECTOR IN A PLANE

$$\Delta \vec{r} = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j}$$

MAGNITUDE OF DISPLACEMENT

$$|\Delta \vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|\Delta \vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

First line: in a plane. Second line: in space (3-D), where the z term is added.

Magnitude of displacement compared with the path length

$\leq$ path length

1.4 Multiplication of a Vector by a Real Number

DEFINITION

Multiplying a vector $\vec{A}$ by a **real number (scalar)** λ gives a **new vector** along the direction of $\vec{A}$. Its magnitude is λ **times** the magnitude of $\vec{A}$.

The answer depends on whether the number is **positive or negative**:

Positive Number λ

Multiplying $\vec{A}$ by a **positive** number λ gives a vector of magnitude $\lambda|\vec{A}|$. It points in the **same direction** as $\vec{A}$.

Direction of $\lambda\vec{A}$

Same as $\vec{A}$

Negative Number $-\lambda$

Multiplying $\vec{A}$ by a **negative** number $-\lambda$ gives a vector of magnitude $\lambda|\vec{A}|$. It points in the **opposite direction** to $\vec{A}$.

Direction of $-\lambda\vec{A}$

Opposite to $\vec{A}$

MAGNITUDE AFTER MULTIPLICATION

$$|\lambda\vec{A}| = \lambda|\vec{A}| \quad \text{and} \quad |-\lambda\vec{A}| = \lambda|\vec{A}|$$

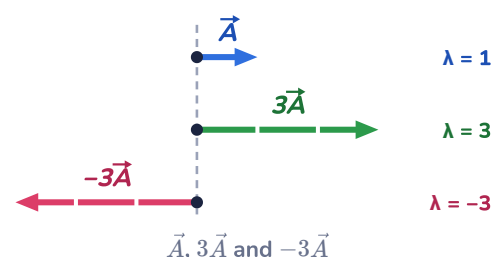
In both cases, the magnitude becomes λ times. For a negative number, only the direction reverses.

EXAMPLES

EXAMPLE

Multiply $\vec{A}$ by $\lambda = 3$. We get $3\vec{A}$. It has the **same direction** as $\vec{A}$, but it is **three times as long**.

Multiply $\vec{A}$ by $\lambda = -3$. We get $-3\vec{A}$. It is also three times as long, but it points in the **opposite direction**.



EXAMPLE

A **constant velocity vector** multiplied by **time** gives a **displacement vector**. This displacement is in the direction of the velocity.

Velocity $\times$ time

Displacement

1.5 Resultant Vector

DEFINITION

A **single vector** that has the **same effect** as two or more vectors acting together is called their **resultant vector**.

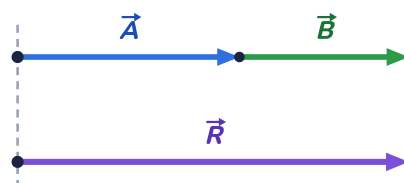
Two simple cases are given below.

• CASE I • VECTORS IN THE SAME DIRECTION

Let $\vec{A}$ and $\vec{B}$ act in the **same direction**. The resultant also points in that direction. Its magnitude is the **sum** of their magnitudes, $(A + B)$.

Resultant vector

$$\vec{R} = \vec{A} + \vec{B}$$



Same direction: $R = A + B$

• CASE II • VECTORS IN OPPOSITE DIRECTIONS

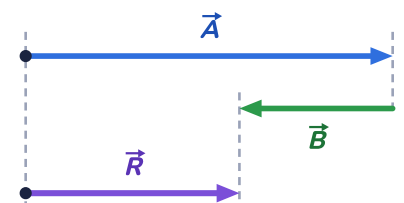
Let $\vec{A}$ and $\vec{B}$ act in **opposite directions**. The resultant points along the vector with the **bigger magnitude**. Its magnitude is the **difference** of their magnitudes, $(A - B)$.

Resultant vector

$$\vec{R} = \vec{A} + \vec{B}$$

Its magnitude (when $A > B$)

$$R = A - B$$



Opposite directions: $R = A - B$, along $\vec{A}$

If $B > A$

$\vec{R}$ is along $\vec{B}$

If $A > B$

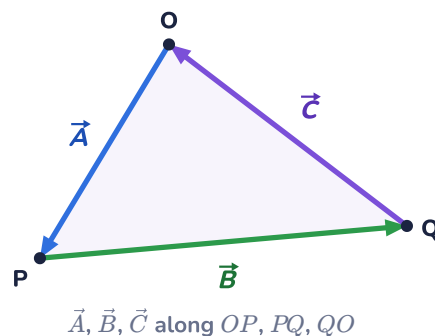
$\vec{R}$ is along $\vec{A}$

● CONDITIONS FOR ZERO RESULTANT VECTOR

Triangle Condition

Three vectors act on a point object. Suppose they can be drawn head to tail to form the **three sides of a triangle**, going round in **one order**. Then they add up to **zero**, and the object stays in **equilibrium**.

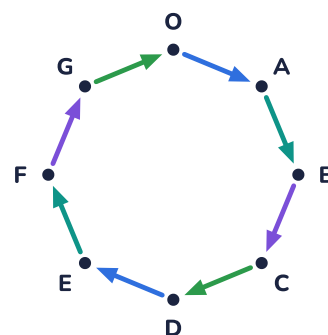
$$\vec{A} + \vec{B} + \vec{C} = \vec{0}$$



Polygon Condition

Any number of vectors drawn head to tail as the **sides of a closed polygon**, in one order, also add up to **zero**. The object is then in equilibrium.

$$\vec{OA} + \vec{AB} + \vec{BC} + \dots + \vec{GO} = \vec{0}$$



● CONDITIONS FOR EQUILIBRIUM

i No Linear Motion

The **net force** acting on the body is **zero**.

ii No Rotational Motion

The **net torque** of all the forces on it is **zero**.

iii Stable Equilibrium

Its **potential energy is the least** (minimum).

1.6 Addition of Vectors – Graphical Method

We can add two vectors only when they are of the **same kind (nature)**. A displacement adds to another displacement. It **never adds to a force**.

Vectors of the same nature (displacement + displacement)

Can be added

Vectors of different nature (displacement + force)

Cannot be added

In the graphical method, we draw the vectors and **see** their resultant. Three laws are used.

• 1 • TRIANGLE LAW OF VECTOR ADDITION

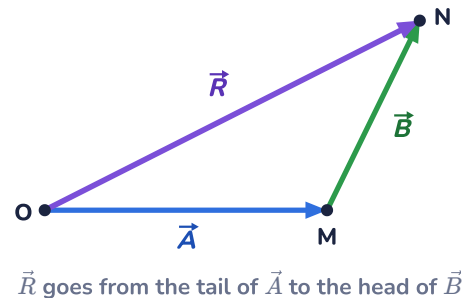
LAW

Place the tail of the second vector at the head of the first. The two vectors now form **two sides of a triangle**. The **third side**, drawn from the tail of the first to the head of the second, is their resultant in **magnitude and direction**.

Draw $\overrightarrow{OM} = \vec{A}$. From the head M , draw $\overrightarrow{MN} = \vec{B}$. The resultant $\overrightarrow{ON}$ joins the tail of $\vec{A}$ to the head of $\vec{B}$.

$$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$$

$$\vec{R} = \vec{A} + \vec{B}$$



• 2 • PARALLELOGRAM LAW OF VECTOR ADDITION

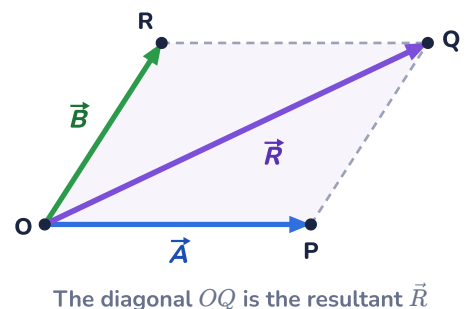
LAW

Draw two vectors from **one common point** as the **two adjacent sides of a parallelogram**. Complete the parallelogram. Its **diagonal** that starts at the common point is the resultant, in **magnitude and direction**.

Draw $\overrightarrow{OP} = \vec{A}$ and $\overrightarrow{OR} = \vec{B}$ from the common point O . Complete the parallelogram $OPQR$. The diagonal $\overrightarrow{OQ}$ is the resultant.

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OR}$$

$$\vec{R} = \vec{A} + \vec{B}$$



• 3 • POLYGON LAW OF VECTOR ADDITION

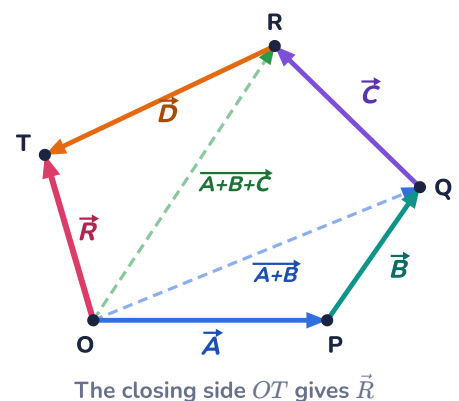
LAW

Draw many vectors **head to tail**. They form the sides of an **open polygon**. The side that **closes the polygon**, drawn from the first tail to the last head, is their resultant in **magnitude and direction**.

Here $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ run head to tail from O to T . So $\overrightarrow{OT}$ is the resultant.

Closing side $\overrightarrow{OT}$

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$



• PROPERTIES OF VECTOR ADDITION

i Commutative Law

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

ii Associative Law

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

iii Distributive Law

$$\lambda(\vec{A} + \vec{B}) = \lambda\vec{A} + \lambda\vec{B}$$

iv Additive Identity

$$\vec{A} + \vec{0} = \vec{A}$$

1.7 Subtraction of Vectors – Graphical Method

DEFINITION

Subtracting $\vec{B}$ from $\vec{A}$ means **adding the negative of $\vec{B}$** , that is $(-\vec{B})$, to $\vec{A}$. So first **invert $\vec{B}$** . Then add it to $\vec{A}$ by the laws of vector addition.

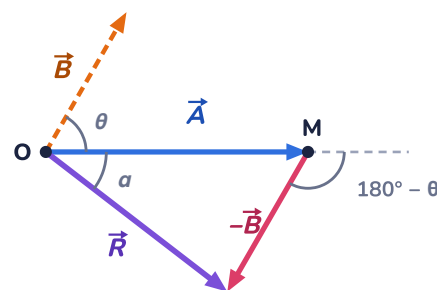
SUBTRACTION OF VECTORS

$$\vec{R} = \vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

• GRAPHICAL METHOD

- ◆ Reverse the direction of $\vec{B}$ to get $-\vec{B}$.
- ◆ Shift $-\vec{B}$ parallel to itself. Place its tail at the head of $\vec{A}$.
- ◆ Join the tail of $\vec{A}$ to the head of $-\vec{B}$. This line is $\vec{R} = \vec{A} - \vec{B}$.

We can also add $-\vec{B}$ to $\vec{A}$ by the **parallelogram law**. The result is the same.



$$\text{Subtraction: } \vec{R} = \vec{A} + (-\vec{B})$$

• PROPERTIES OF VECTOR SUBTRACTION

i Not Commutative

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$$

ii Not Associative

$$\vec{A} - (\vec{B} - \vec{C}) \neq (\vec{A} - \vec{B}) - \vec{C}$$

iii Distributive

$$\lambda(\vec{A} - \vec{B}) = \lambda\vec{A} - \lambda\vec{B}$$

1.8 Resolution of a Vector in a Plane

DEFINITION

Resolution of a vector means breaking one vector into two or more vectors along different directions. These parts, acting together, have the **same effect** as the original vector.

Components of a Vector

The parts obtained by resolving a vector are called its **components**.

Oblique Components

When a vector is resolved along the directions of **two given vectors**, the two parts are called **oblique components**.

To resolve a vector, we write it with **unit vectors**. Any vector $\vec{r}$ in a plane can be written as a **linear combination** of the unit vectors $\hat{i}$ and $\hat{j}$. These two unit vectors are perpendicular to each other.

VECTOR IN TERMS OF COMPONENTS

$$\vec{r} = x\hat{i} + y\hat{j}$$

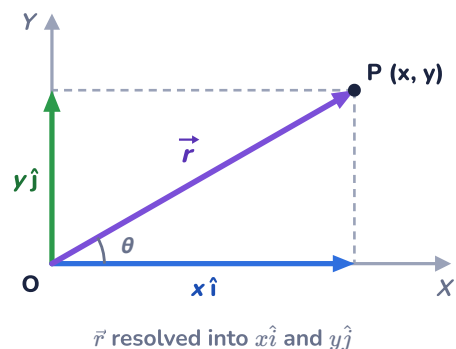
$x\hat{i}$ and $y\hat{j}$ are the **perpendicular components** of $\vec{r}$. x and y are the **resolved parts** of $\vec{r}$ along the x-axis and the y-axis.

MAGNITUDE

$$|\vec{r}| = \sqrt{x^2 + y^2}$$

DIRECTION (ANGLE WITH X-AXIS)

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

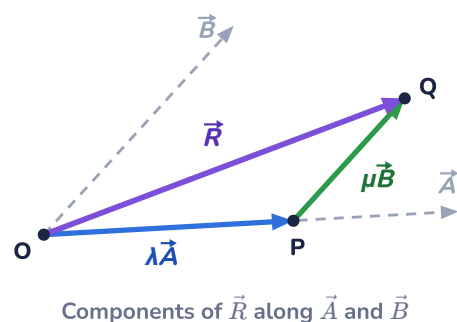


• RESOLVING A VECTOR ALONG TWO GIVEN DIRECTIONS

Let $\vec{OQ}$ stand for $\vec{R}$. The triangle law gives:

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

Here $\vec{OP}$ and $\vec{PQ}$ are the components of $\vec{R}$ along $\vec{A}$ and $\vec{B}$. Take $\vec{OP} = \lambda \vec{A}$ and $\vec{PQ} = \mu \vec{B}$, where λ and μ are real numbers. Then $\vec{R}$ becomes a linear combination of $\vec{A}$ and $\vec{B}$.



RESULTANT ALONG TWO GIVEN DIRECTIONS

$$\vec{R} = \lambda \vec{A} + \mu \vec{B}$$

• RECTANGULAR COMPONENTS

DEFINITION

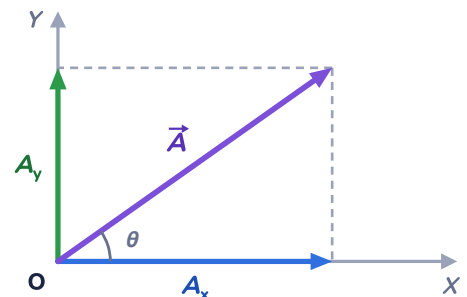
The two components of a vector that are **at right angles** to each other are called its **rectangular components**.

MAGNITUDE

$$A = \sqrt{A_x^2 + A_y^2}$$

DIRECTION (ANGLE WITH X-AXIS)

$$\theta = \tan^{-1} \left(\frac{A_y}{A_x} \right)$$



Rectangular components of $\vec{A}$

NOTE

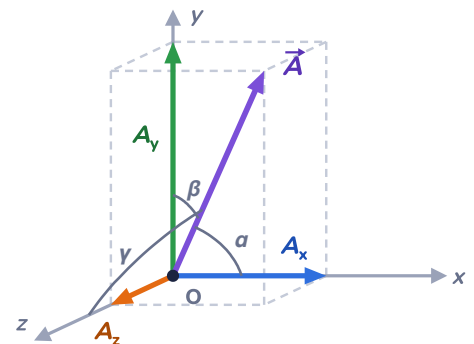
Here A_x and A_y are the components of $\vec{A}$ along the **x-axis** ($\hat{i}$) and the **y-axis** ($\hat{j}$).

1.9 Resolution of a Vector in Space

In the same way, a vector $\vec{A}$ in **three dimensions** (space) can be resolved into **three components**. These are along the x, y and z axes.

Let α , β and γ be the angles that $\vec{A}$ makes with the **x, y and z axes**. Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along these axes.

Resolving $\vec{A}$ gives its three components.



Components of $\vec{A}$ in space

X-COMPONENT

$$A_x = A \cos \alpha$$

Y-COMPONENT

$$A_y = A \cos \beta$$

Z-COMPONENT

$$A_z = A \cos \gamma$$

VECTOR IN THREE DIMENSIONS

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

MAGNITUDE

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

POSITION VECTOR IN SPACE

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

• DIRECTION COSINES

DEFINITION

The cosines of the angles α , β and γ that a vector makes with the x, y and z axes are called its **direction cosines**. They are written as l , m and n .

WITH X-AXIS

$$\cos \alpha = \frac{A_x}{A} = l$$

WITH Y-AXIS

$$\cos \beta = \frac{A_y}{A} = m$$

WITH Z-AXIS

$$\cos \gamma = \frac{A_z}{A} = n$$

DERIVATION

Put $A_x = A \cos \alpha$, $A_y = A \cos \beta$ and $A_z = A \cos \gamma$ in $A^2 = A_x^2 + A_y^2 + A_z^2$:

1

$$A^2 = A^2 \cos^2 \alpha + A^2 \cos^2 \beta + A^2 \cos^2 \gamma$$

2

Take A^2 common:

$$A^2 = A^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$$

3

Divide both sides by A^2 :

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

PROPERTY OF DIRECTION COSINES

$$l^2 + m^2 + n^2 = 1$$

Add the squares of the three direction cosines of any vector. The total is always **one**.

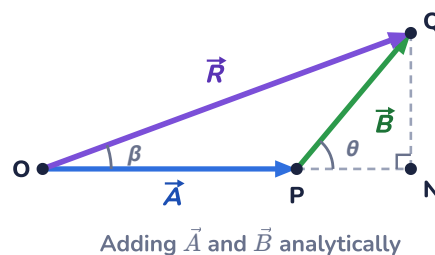
NOTE

α , β and γ are **space angles**. The pairs of lines that form them do **not lie in one plane**.

1.10 Addition of Vectors – Analytical Method

SETUP – GIVEN INFORMATION

- Two vectors $\vec{A}$ and $\vec{B}$ act on a particle. The angle between them is θ .
- They are drawn head to tail as the sides $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ of $\triangle OPQ$.
- The closing side $\overrightarrow{OQ}$ is then their resultant $\vec{R}$ (triangle law).
- Draw $QN \perp OP$ produced.



DERIVATION

- In the right triangle QPN , the side $PQ = B$. So:

$$PN = B \cos \theta \quad \dots (i) \quad \text{and} \quad QN = B \sin \theta \quad \dots (ii)$$

- In the right triangle ONQ , $OQ^2 = ON^2 + NQ^2 = (OP + PN)^2 + (QN)^2$. Hence:

$$R^2 = (A + B \cos \theta)^2 + (B \sin \theta)^2$$

- Expand the brackets and use $\cos^2 \theta + \sin^2 \theta = 1$:

$$R^2 = A^2 + B^2 \cos^2 \theta + 2AB \cos \theta + B^2 \sin^2 \theta = A^2 + B^2 + 2AB \cos \theta$$

MAGNITUDE OF THE RESULTANT

$$R = |\vec{R}| = \sqrt{A^2 + B^2 + 2AB \cos \theta}$$

Let $\vec{R}$ make an angle β with the direction of $\vec{A}$. In the right triangle QNO :

$$\tan \beta = \frac{QN}{ON} = \frac{QN}{OP + PN} = \frac{B \sin \theta}{A + B \cos \theta}$$

DIRECTION OF THE RESULTANT

$$\beta = \tan^{-1} \left(\frac{B \sin \theta}{A + B \cos \theta} \right)$$

• SPECIAL CASES FOR THE MAGNITUDE OF R

$$\theta = 0^\circ$$

$$R = A + B$$

This is the **maximum** value.

$$\theta = 90^\circ$$

$$R = \sqrt{A^2 + B^2}$$

The two vectors are **perpendicular**.

$$\theta = 180^\circ$$

$$R = A - B$$

This is the **minimum** value.

• MORE THAN TWO VECTORS

The same method works for **any number of vectors**. Take three of them:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}, \quad \vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}, \quad \vec{c} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

Their resultant $\vec{r} = \vec{a} + \vec{b} + \vec{c}$ has these components:

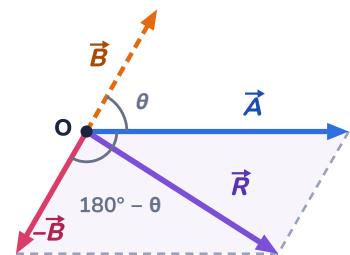
$$r_x = a_x + b_x + c_x, \quad r_y = a_y + b_y + c_y, \quad r_z = a_z + b_z + c_z$$

1.11 Subtraction of Vectors – Analytical Method

Let $\vec{A}$ and $\vec{B}$ make an angle θ with each other. To subtract $\vec{B}$ from $\vec{A}$, we first **invert** $\vec{B}$. Then we add it to $\vec{A}$.

Resultant vector

$$\vec{R} = \vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$



$\vec{R} = \vec{A} - \vec{B}$ by the parallelogram method

DERIVATION

- 1 The angle between $\vec{A}$ and the inverted vector $(-\vec{B})$ is $(180^\circ - \theta)$. Use the formula for addition:

$$R = \sqrt{A^2 + B^2 + 2AB \cos(180^\circ - \theta)}$$

- 2 Since $\cos(180^\circ - \theta) = -\cos \theta$, the magnitude of the difference becomes:

$$R = \sqrt{A^2 + B^2 - 2AB \cos \theta}$$

MAGNITUDE OF THE DIFFERENCE

$$R = \sqrt{A^2 + B^2 - 2AB \cos \theta}$$

where, θ = angle between $\vec{A}$ and $\vec{B}$

• SPECIAL CASES FOR THE MAGNITUDE OF R

$$\theta = 0^\circ$$

$$R = A - B$$

This is the **minimum** value.

$$\theta = 90^\circ$$

$$R = \sqrt{A^2 + B^2}$$

The two vectors are **perpendicular**.

$$\theta = 180^\circ$$

$$R = A + B$$

This is the **maximum** value.

• SUM AND DIFFERENCE SIDE BY SIDE

MAGNITUDE OF THE SUM

$$R = \sqrt{A^2 + B^2 + 2AB \cos \theta}$$

MAGNITUDE OF THE DIFFERENCE

$$R = \sqrt{A^2 + B^2 - 2AB \cos \theta}$$

1.12 Scalar Product (Dot Product)

DEFINITION

The **scalar product** (dot product) of $\vec{A}$ and $\vec{B}$ is found by multiplying their **magnitudes** by the **cosine of the angle** θ between them. Its value is a **scalar**.

SCALAR PRODUCT

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

• GEOMETRICAL MEANING • PROJECTION

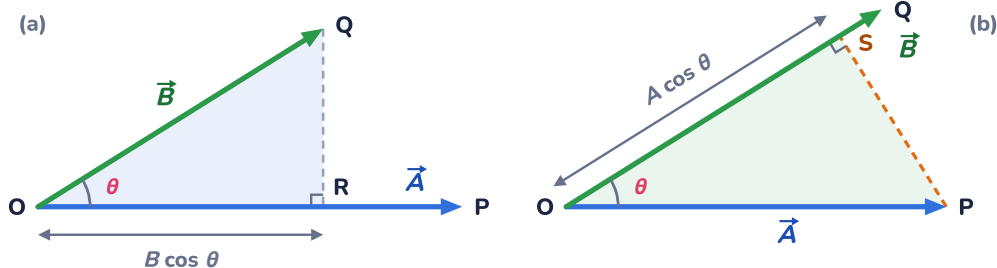
In other words, $\vec{A} \cdot \vec{B} = (\text{size of one vector}) \times (\text{projection of the second vector on the first})$.

(a) $B \cos \theta$

Projection of $\vec{B}$ on $\vec{A}$

(b) $A \cos \theta$

Projection of $\vec{A}$ on $\vec{B}$



The scalar product as a projection

• SPECIAL CASES

I Parallel

Here $\theta = 0^\circ$.

$$\vec{A} \cdot \vec{B} = AB$$

II Perpendicular

Here $\theta = 90^\circ$.

$$\vec{A} \cdot \vec{B} = 0$$

III Anti-parallel

Here $\theta = 180^\circ$.

$$\vec{A} \cdot \vec{B} = -AB$$

• DOT PRODUCT IN TERMS OF COMPONENTS

Write both vectors in component form. Then use the distributive law:

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

Same Unit Vectors

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

Perpendicular Unit Vectors

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

SCALAR PRODUCT IN COMPONENT FORM

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

• PROPERTIES OF THE DOT PRODUCT

i Self Product

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

ii Commutative Law

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

iii Distributive Law

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

iv Scalar Multiple

$$(\lambda \vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda \vec{b})$$

1.13 Vector Product (Cross Product)

DEFINITION

The **vector product** (cross product) of $\vec{A}$ and $\vec{B}$ is a **vector**. Its magnitude is A times B times the **sine of the angle** θ between them.

VECTOR PRODUCT

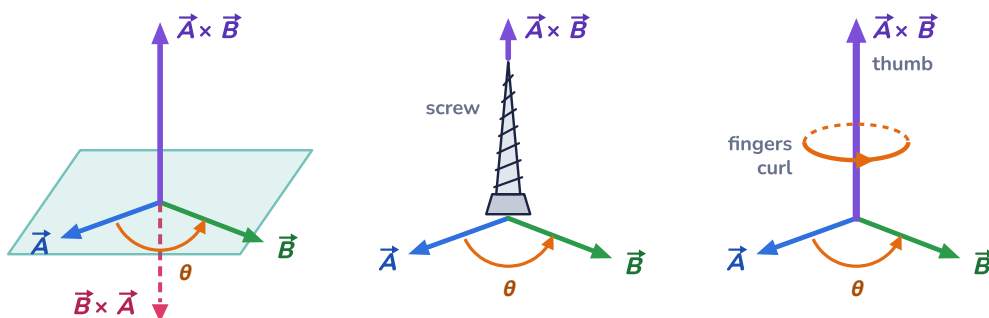
$$\vec{A} \times \vec{B} = AB \sin \theta \hat{n}$$

where, $\hat{n}$ = unit vector **perpendicular to the plane** of $\vec{A}$ and $\vec{B}$, in the direction of $\vec{A} \times \vec{B}$

PRINCIPLE

Right-Hand Rule

Hold the **right hand** so that its fingers turn from $\vec{A}$ towards $\vec{B}$. The **thumb** then gives the direction of both $\hat{n}$ and $\vec{A} \times \vec{B}$.



$\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$. A screw turned from $\vec{A}$ to $\vec{B}$, or the right thumb, points along it

CROSS PRODUCT IN TERMS OF COMPONENTS

Let $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$. Then:

DETERMINANT FORM

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x) + \hat{k}(a_x b_y - a_y b_x)$$

Same Unit Vectors

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

Cyclic Unit Vectors

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

SPECIAL CASES

Parallel or Anti-parallel

$$\vec{A} \times \vec{B} = \vec{0}$$

Perpendicular

Here $\theta = 90^\circ$.

$$|\vec{A} \times \vec{B}| = AB$$

Area of a Triangle

$$\text{Area} = \frac{1}{2} |\vec{A} \times \vec{B}|$$

$\vec{A}$ and $\vec{B}$ are two sides of the triangle.

PROPERTIES OF THE CROSS PRODUCT

i Not Commutative (Anti-commutative)

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

ii Self Product

$$\vec{a} \times \vec{a} = \vec{0}$$

iii Distributive Law

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

iv Scalar Multiple

$$(\lambda \vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}) = \vec{a} \times (\lambda \vec{b})$$

• UNIT VECTORS ON A CIRCLE

Place $\hat{i}$, $\hat{j}$ and $\hat{k}$ on a circle. The cross product of **two neighbouring unit vectors taken anticlockwise** gives the **third unit vector**. Taken clockwise, it gives the negative of the third one.

Anticlockwise

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

Clockwise (reverse order)

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$$

02 PART 2 Motion in a Plane

Vectors now help us study **two-dimensional motion**. With them, we describe the **position**, **displacement**, **velocity** and **acceleration** of an object.

2.1 Position, Displacement and Velocity Vectors

• POSITION VECTOR

DEFINITION

The vector drawn from a **reference point** to the position of a particle is called its **position vector**.

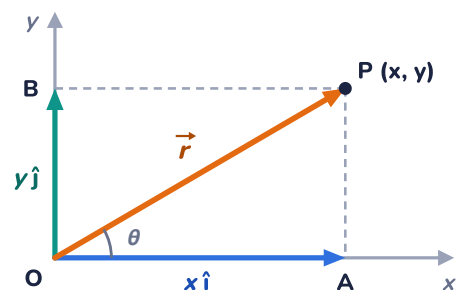
For a particle at P in the xy -plane, the parallelogram law gives $\vec{OP} = \vec{OA} + \vec{OB}$.

POSITION VECTOR IN A PLANE

$$\vec{r} = x\hat{i} + y\hat{j}$$

POSITION VECTOR IN SPACE

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



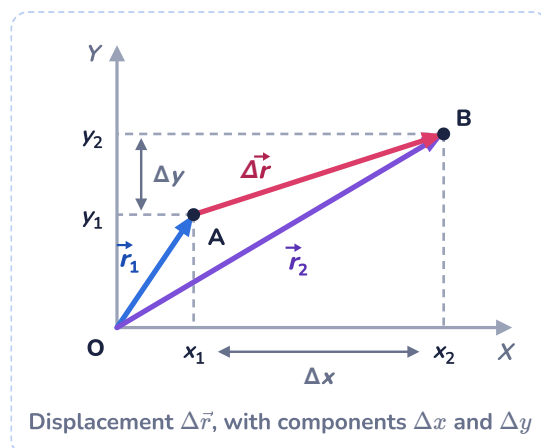
Position vector $\vec{r}$ of a particle at $P(x, y)$

• DISPLACEMENT

A particle moves from A to B . At time t_1 it is at A , with position vector $\vec{r}_1$. At time t_2 it is at B , with position vector $\vec{r}_2$.

By the triangle law, $\vec{OA} + \vec{AB} = \vec{OB}$. So the displacement is:

$$\vec{AB} = \vec{OB} - \vec{OA} = \vec{r}_2 - \vec{r}_1$$



DISPLACEMENT IN A PLANE AND IN SPACE

$$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$\Delta \vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

First line: in a plane. Second line: in space.

• VELOCITY

DEFINITION

Velocity is the rate at which the displacement of an object changes with time, in a given direction. It has two forms: **average velocity** and **instantaneous velocity**.

• AVERAGE VELOCITY

DEFINITION

The **average velocity** of an object is its **displacement** divided by the **time interval** taken for it.

AVERAGE VELOCITY

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

In component form, the average velocity, its magnitude and its direction are:

COMPONENT FORM

$$\vec{v}_{av} = v_x\hat{i} + v_y\hat{j}$$

MAGNITUDE

$$v_{av} = \sqrt{v_x^2 + v_y^2}$$

DIRECTION

$$\theta = \tan^{-1} \left(\frac{v_y}{v_x} \right)$$

2.2 Instantaneous Velocity

DEFINITION

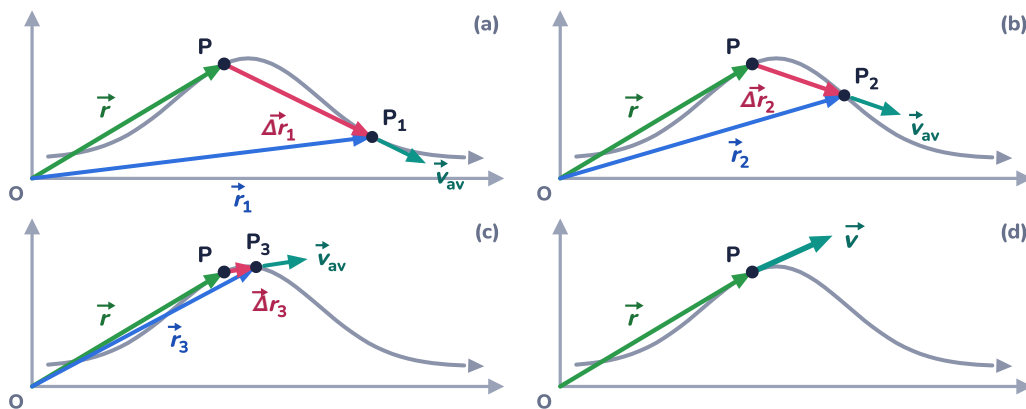
The velocity of an object at a particular **instant of time** (t) is called its **instantaneous velocity**. It is the limit of the average velocity as Δt approaches zero.

INSTANTANEOUS VELOCITY

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

• UNDERSTANDING THE LIMITING PROCESS

The curve shows the path of the object. At time t , the object is at point P . P_1 , P_2 and P_3 are its positions after the time intervals Δt_1 , Δt_2 and Δt_3 . Here $\Delta t_1 > \Delta t_2 > \Delta t_3$.



As Δt becomes smaller, the chord $\Delta \vec{r}$ turns towards the tangent at P

(a) Δt_1 average $\vec{v}$

(b) Δt_2 average $\vec{v}$

(c) Δt_3 average $\vec{v}$

(d) $\Delta t \rightarrow 0$ instantaneous $\vec{v}$

When Δt becomes very small (tends to zero), the average velocity becomes the velocity $\vec{v}$ at P . Then $\vec{v}$ lies along the **tangent to the path**.

NOTE

The **instantaneous velocity** at any point is along the **tangent** to the path there. It shows the **direction of motion** of the object at that point.

2.3 Acceleration in a Plane

DEFINITION

Acceleration is the **change in velocity** divided by the **time taken** for that change. It has two forms: **average** and **instantaneous** acceleration.

ACCELERATION

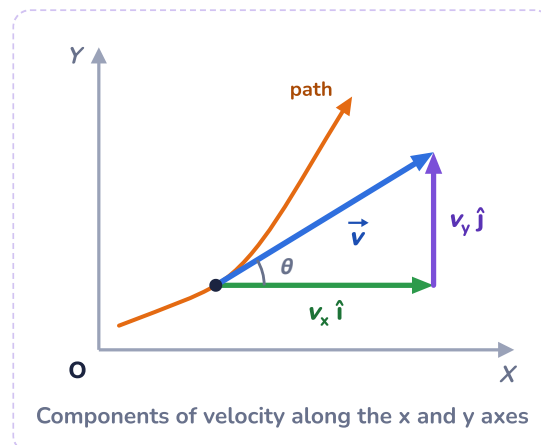
$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

• AVERAGE ACCELERATION

The **average acceleration** is the change in velocity $\Delta \vec{v}$ divided by the time interval Δt :

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\Delta v_x \hat{i} + \Delta v_y \hat{j}}{\Delta t} = \frac{\Delta v_x}{\Delta t} \hat{i} + \frac{\Delta v_y}{\Delta t} \hat{j}$$

The figure shows a velocity $\vec{v}$ split into its components $v_x \hat{i}$ and $v_y \hat{j}$.



AVERAGE ACCELERATION IN COMPONENT FORM

$$\vec{a}_{av} = a_x \hat{i} + a_y \hat{j}$$

2.4 Instantaneous Acceleration

DEFINITION

Instantaneous acceleration is the **limiting value** of the average acceleration when the time interval **tends to zero**.

INSTANTANEOUS ACCELERATION

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

In component form, the instantaneous acceleration and its magnitude are:

COMPONENT FORM

$$\vec{a} = a_x \hat{i} + a_y \hat{j}$$

MAGNITUDE

$$a = \sqrt{a_x^2 + a_y^2}$$

Each component is the rate of change of the matching velocity component. In terms of x and y :

X-COMPONENT

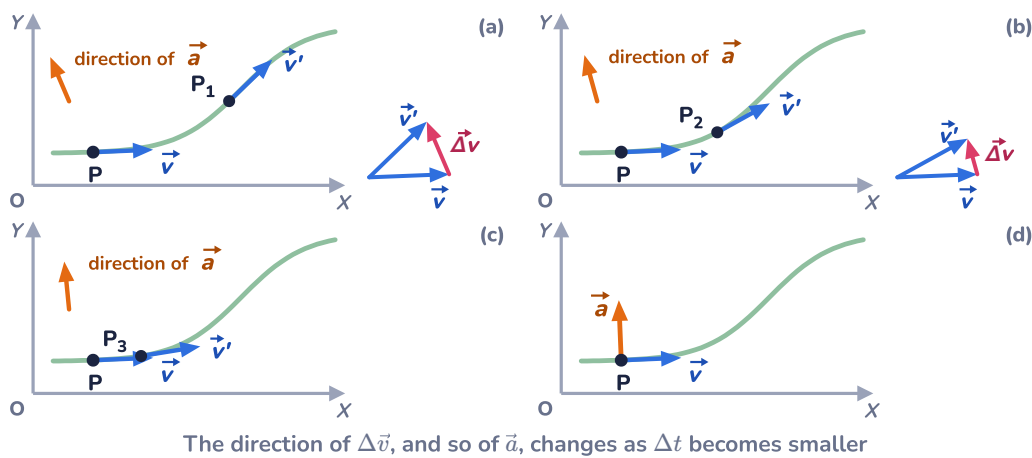
$$a_x = \frac{dv_x}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2}$$

Y-COMPONENT

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt} \left(\frac{dy}{dt} \right) = \frac{d^2y}{dt^2}$$

• UNDERSTANDING THE LIMITING PROCESS

At time t , the object is at point P . P_1 , P_2 and P_3 are its positions after the time intervals Δt_1 , Δt_2 and Δt_3 , where $\Delta t_1 > \Delta t_2 > \Delta t_3$. The figure also shows the velocity vectors at these points. We find $\Delta \vec{v}$ by the **triangle law**. The direction of $\vec{a}$ is **parallel to $\Delta \vec{v}$** .



(a) Δt_1 average $\vec{a}$

(b) Δt_2 average $\vec{a}$

(c) Δt_3 average $\vec{a}$

(d) $\Delta t \rightarrow 0$ instantaneous $\vec{a}$

Going from (a) to (d), Δt gets smaller and $\Delta \vec{v}$ turns. So $\vec{a}$ turns too. In (d), $\Delta t \rightarrow 0$, and the average acceleration becomes the **instantaneous acceleration**. We simply call it **acceleration**.

NOTE

In a **plane** or in **space**, the angle between $\vec{v}$ and $\vec{a}$ can be **anything from 0° to 180°** . In **one-dimensional** motion, they always lie on **one straight line**. They are either along each other or opposite to each other.

2.5 Motion in a Plane with Uniform Velocity

DEFINITION

Uniform velocity means that a body has **equal displacements in equal time intervals**, however short the intervals.

Let an object move with uniform velocity $\vec{v}$ in the xy-plane. Let its position vector be $\vec{r}(0)$ at $t = 0$ and $\vec{r}(t)$ at time t . Then:

POSITION-TIME RELATION

$$\vec{v} = \frac{\vec{r}(t) - \vec{r}(0)}{t - 0} \Rightarrow \vec{r}(t) = \vec{r}(0) + \vec{v} t$$

In terms of rectangular components:

$$\vec{v} = v_x \hat{i} + v_y \hat{j}, \quad v = \sqrt{v_x^2 + v_y^2}$$

$$\vec{r}(0) = x(0)\hat{i} + y(0)\hat{j}, \quad \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

ALONG X-AXIS

$$x(t) = x(0) + v_x t$$

ALONG Y-AXIS

$$y(t) = y(0) + v_y t$$

CONCEPT

So uniform motion in a plane is made of **two independent uniform motions**, one along x and one along y. These two directions are **perpendicular** to each other.

2.6 Motion in a Plane with Constant Acceleration

DEFINITION

Uniform (constant) acceleration means that the velocity of a body changes by **equal amounts in equal time intervals**, however short the intervals.

SETUP – GIVEN INFORMATION

- ◆ An object moves in the xy-plane with **constant acceleration** $\vec{a}$.
- ◆ Its velocity is $\vec{v}_0$ at $t = 0$ and $\vec{v}$ at time t .
- ◆ Its position vectors are $\vec{r}_0$ at $t = 0$ and $\vec{r}$ at time t .

• VELOCITY-TIME RELATION

DERIVATION

1 From the definition of **average acceleration**:

$$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t - 0} = \frac{\vec{v} - \vec{v}_0}{t}$$

2 Rearrange the terms:

$$\vec{v} = \vec{v}_0 + \vec{a} t$$

VELOCITY-TIME RELATION

$$\vec{v} = \vec{v}_0 + \vec{a} t$$

Written for the x and y directions separately:

X-COMPONENT OF VELOCITY

$$v_x = v_{0x} + a_x t$$

Y-COMPONENT OF VELOCITY

$$v_y = v_{0y} + a_y t$$

RESULT

Each rectangular component of velocity changes with time exactly like the velocity in **one-dimensional uniformly accelerated motion**.

• POSITION-TIME RELATION

DERIVATION

1 For constant acceleration, the **average velocity** is:

$$\vec{v}_{av} = \frac{\vec{v}_0 + \vec{v}}{2}$$

2 Displacement = average velocity × time interval:

$$\vec{r} - \vec{r}_0 = \left(\frac{\vec{v} + \vec{v}_0}{2} \right) t = \left[\frac{(\vec{v}_0 + \vec{a}t) + \vec{v}_0}{2} \right] t$$

3 Simplify:

$$\vec{r} - \vec{r}_0 = \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

4 So the **position-time relation** is:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

POSITION-TIME RELATION

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

In terms of rectangular components:

$$x\hat{i} + y\hat{j} = x_0\hat{i} + y_0\hat{j} + (v_{0x}\hat{i} + v_{0y}\hat{j})t + \frac{1}{2}(a_x\hat{i} + a_y\hat{j})t^2$$

Equate the coefficients of $\hat{i}$ and $\hat{j}$ on both sides:

ALONG X-AXIS

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

ALONG Y-AXIS

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

NOTE

So two-dimensional motion with constant acceleration is just **two one-dimensional motions** happening together. One is along x and the other along y, at **right angles** to each other.

03

PART 3

Projectile and Circular Motion

Here we study two important motions in a plane. A **projectile** moves under gravity with a **constant acceleration**. A body in **uniform circular motion** moves on a circle with a **constant speed**.

3.1 Projectile Motion

DEFINITION

Projectile motion is the motion of an object that is thrown near the Earth's surface with some **initial velocity** and then moves **only under gravity**. Its path is a **curve**.

Such a moving object is called a **projectile**. Its curved path is called its **trajectory**.

EXAMPLES OF PROJECTILES

- ◆ A **tennis ball** or **baseball** moving through the air.
- ◆ A **bullet** after it leaves a rifle.
- ◆ An object let go from the **window of a moving train**.
- ◆ **Water** shooting out from a hole in the side of a tank.
- ◆ A **javelin** after an athlete throws it.

ASSUMPTIONS MADE IN THIS STUDY

- ◆ **Air resistance** is ignored (there is no friction due to air).
- ◆ The Earth's **rotation** and its **curvature** have a negligible effect.

- ◆ The **acceleration due to gravity** (g) has the same size and direction everywhere along the path.

MATHEMATICAL ANALYSIS

Throw an object from O with speed u , at an angle θ above the horizontal. Split u into two parts:

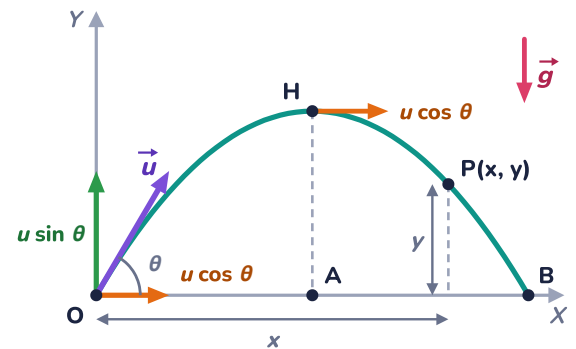
Horizontal (along OX): $u \cos \theta$

Stays constant

Vertical (along OY): $u \sin \theta$

Decreases

The vertical component becomes **zero at the highest point H** . At H , the velocity is fully **horizontal**.



A projectile thrown at angle θ with the horizontal

3.2 Equation of Path of a Projectile

SETUP – GIVEN INFORMATION

The object reaches the point $P(x, y)$ at time t . We treat its motion as two motions along perpendicular directions:

- ◆ **Horizontal (OX):** $a_x = 0$, $u_x = u \cos \theta$. The horizontal velocity stays constant.
- ◆ **Vertical (OY):** $a_y = -g$, $u_y = u \sin \theta$. Gravity slows down the upward motion.

DERIVATION

- Horizontal motion:** the distance covered in time t is:

$$x = u \cos \theta \cdot t \Rightarrow t = \frac{x}{u \cos \theta} \quad \dots (i)$$

- Vertical motion:** use $y = u_y t + \frac{1}{2} a_y t^2$ with $u_y = u \sin \theta$ and $a_y = -g$:

$$y = u \sin \theta \cdot t - \frac{1}{2} g t^2$$

- Put the value of t from Eq. (i):

$$y = u \sin \theta \left(\frac{x}{u \cos \theta} \right) - \frac{1}{2} g \left(\frac{x}{u \cos \theta} \right)^2$$

- Simplify to get the **equation of the path**:

$$y = x \tan \theta - \frac{g x^2}{2 u^2 \cos^2 \theta}$$

$$y = x \tan \theta - \frac{g x^2}{2u^2 \cos^2 \theta}$$

RESULT

This equation has the form $y = ax - bx^2$. It is the equation of a **parabola**. So the **path of a projectile is a parabola**.

3.3 Time of Flight

DEFINITION

The **total time** for which a projectile stays in flight is called its **time of flight** (T).

It is the time from O to H (**time of ascent**) plus the time from H to B (**time of descent**). So $T = 2t$, where $t = T/2$ is the time of ascent.

DERIVATION

- 1 At the highest point H , the vertical velocity is zero. Here $u_y = u \sin \theta$, $a_y = -g$ and $t = T/2$. Use $v_y = u_y + a_y t$:

$$0 = u \sin \theta - g \cdot \frac{T}{2}$$

- 2 Solve for T :

$$\frac{T}{2} = \frac{u \sin \theta}{g} \Rightarrow T = \frac{2u \sin \theta}{g}$$

TIME OF FLIGHT

$$T = \frac{2u \sin \theta}{g}$$

Time to rise (**time of ascent**) = time to fall (**time of descent**)

$$\frac{u \sin \theta}{g}$$

3.4 Maximum Height

DEFINITION

The **greatest vertical height** reached by a projectile above the point of projection is called its **maximum height** (H).

DERIVATION

- 1 Take the upward motion from O to H . Here $u_y = u \sin \theta$, $a_y = -g$ and $v_y = 0$. Use $v_y^2 = u_y^2 + 2a_y H$:

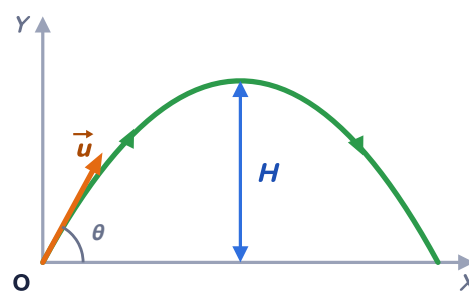
$$0 = (u \sin \theta)^2 - 2gH$$

- 2 Solve for H :

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

MAXIMUM HEIGHT

$$H = \frac{u^2 \sin^2 \theta}{2g}$$



Maximum height H of a projectile

3.5 Horizontal Range

DEFINITION

The **horizontal distance** covered by a projectile during its time of flight is called its **horizontal range** (R).

DERIVATION

- 1 Range = constant horizontal velocity $\times$ time of flight:

$$R = u \cos \theta \times T = u \cos \theta \times \frac{2u \sin \theta}{g}$$

- 2 Use $2 \sin \theta \cos \theta = \sin 2\theta$:

$$R = \frac{u^2 (2 \sin \theta \cos \theta)}{g} = \frac{u^2 \sin 2\theta}{g}$$

HORIZONTAL RANGE

$$R = \frac{u^2 \sin 2\theta}{g}$$

Condition for Maximum Range

R is maximum when $\sin 2\theta = 1$. That is, $2\theta = 90^\circ \Rightarrow \theta = 45^\circ$.

Maximum Range

$$R_{max} = \frac{u^2}{g}$$

3.6 Projectile Thrown at an Angle to the Vertical

Let a projectile be fired at an angle θ with the **vertical**. Then it makes an angle $(90^\circ - \theta)$ with the horizontal.

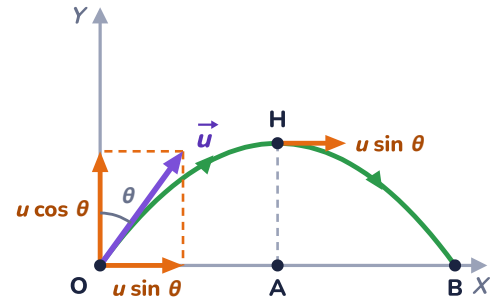
Horizontal component

$$u \sin \theta$$

Vertical component

$$u \cos \theta$$

So we use $\theta \rightarrow (90^\circ - \theta)$ in the formulas above.



Projectile fired at angle θ with the vertical

TIME OF FLIGHT

$$T = \frac{2u \cos \theta}{g}$$

MAXIMUM HEIGHT

$$H = \frac{u^2 \cos^2 \theta}{2g}$$

HORIZONTAL RANGE

$$R = \frac{u^2 \sin 2\theta}{g}$$

3.7 Uniform Circular Motion

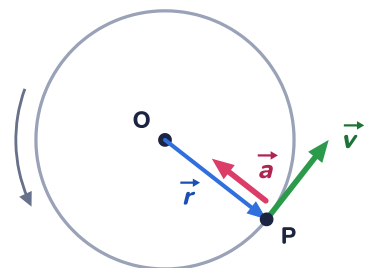
DEFINITION

Uniform circular motion is the motion of an object along a **circle at constant speed**. Here "uniform" tells us that the **speed** does not change.

REMEMBER

Why Is It Accelerated?

The speed is constant, but the particle still **accelerates**. This is because the **direction of its velocity changes** at every point of the circular path.



$\vec{v}$ is along the tangent. $\vec{a}$ points towards the centre O

EXAMPLES

The tip of a clock's **second hand** as it goes round.

Any point on the **rim of a wheel** that turns at a steady rate.

3.8 Terms Related to Circular Motion

1 Angular Displacement

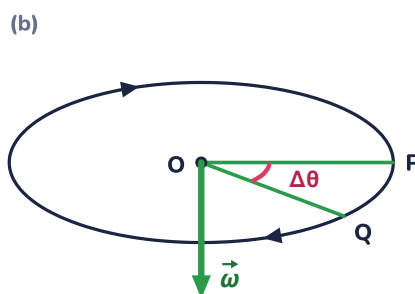
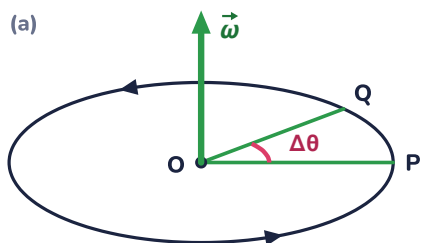
The **angular displacement** ($\Delta\theta$) is the **angle through which the radius vector turns** at the centre in a given time.

It is **dimensionless** and it is a **vector** quantity. Its unit is the **radian**. Its direction is given by the **right-hand rule**.

2 Angular Velocity

The **time rate of change of angular position** is called the **angular velocity**. It is a **vector**. Its unit is **rad s⁻¹** and its dimensions are **[T⁻¹]**.

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



(a) Anticlockwise rotation: $\vec{\omega}$ points up. (b) Clockwise rotation: $\vec{\omega}$ points down (right-hand rule)

3 Time Period (T)

The time taken to complete **one revolution** is called the **time period**. Its unit is the **second**.

4 Frequency (f)

The number of **revolutions completed per unit time** is called the **frequency**. Its unit is the **hertz (Hz)**.

5 Angular Acceleration

The **time rate of change of angular velocity** is called the **angular acceleration**. Unit: **rad s⁻²**. Dimensions: **[T⁻²]**.

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

• IMPORTANT RELATIONS

TIME PERIOD AND FREQUENCY

$$T = \frac{1}{f}$$

ANGULAR VELOCITY AND FREQUENCY

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi f$$

NOTE

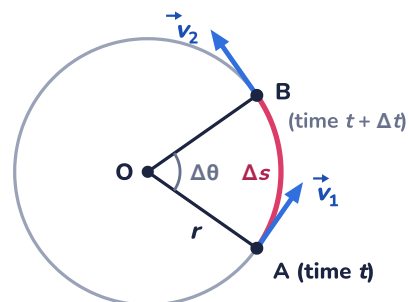
Why These Relations Hold

In **one complete revolution**, the radius vector turns through $\theta = 2\pi$ radian in time $t = T$. So $\omega = \frac{2\pi}{T} = 2\pi f$. Also, one revolution takes $\frac{1}{f}$ second. So the time period is $T = \frac{1}{f}$.

3.9 Relation between Linear and Angular Velocity

A particle moves on a circular track of radius r . In time Δt , it turns through an angle $\Delta\theta$. It covers an arc of length Δs along the track.

The linear velocity $\vec{v}$ is along the **tangent** to the path. The angular velocity ω is along the **axis of rotation**.



The particle moves from A to B in time Δt

DERIVATION

1 Angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t} \quad \dots (i)$$

2 Linear velocity:

$$v = \frac{\Delta s}{\Delta t} \quad \dots (ii)$$

3 Angle = $\frac{\text{arc}}{\text{radius}}$. So the arc length is:

$$\Delta s = r \Delta\theta \quad \dots (iii)$$

4 From (i), (ii) and (iii):

$$v = \frac{\Delta s}{\Delta t} = r \frac{\Delta\theta}{\Delta t} = r\omega$$

$$v = r \frac{\Delta\theta}{\Delta t} = r\omega$$

where, r = radius of the circle

3.10 Centripetal Acceleration

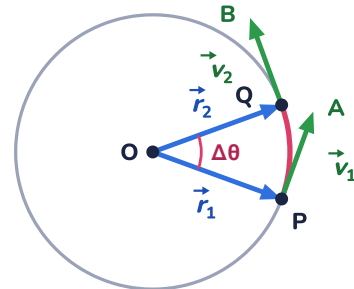
DEFINITION

Centripetal acceleration is the acceleration of a body in uniform circular motion. It always points **towards the centre** of the circle.

The velocity of the particle keeps changing its direction. So the particle has an acceleration that points **radially inwards**.

SETUP – GIVEN INFORMATION

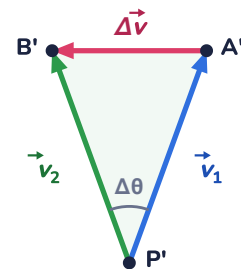
- ◆ A particle of mass m moves on a circle of radius r and centre O . Its speed v is constant. Its angular velocity ω is uniform.
- ◆ At time t it is at P , with $\overrightarrow{OP} = \vec{r}_1$. At time $t + \Delta t$ it is at Q , with $\overrightarrow{OQ} = \vec{r}_2$. Here $\angle POQ = \Delta\theta$ and $|\vec{r}_1| = |\vec{r}_2| = r$.
- ◆ Angular speed of the particle: $\omega = \frac{\Delta\theta}{\Delta t} \dots (i)$
- ◆ The velocities $\vec{v}_1$ (along the tangent PA) and $\vec{v}_2$ (along the tangent QB) have equal magnitudes: $|PA| = |QB| = |\vec{v}|$.



The velocities at P and Q are along the tangents

Now draw the two velocity vectors from one point P' . They form a **triangle of vectors**:

- ◆ $\overrightarrow{P'A'} = \vec{v}_1$ and $\overrightarrow{P'B'} = \vec{v}_2$.
- ◆ The angle between them is also $\Delta\theta$.
- ◆ The side $\overrightarrow{A'B'}$ gives the change in velocity.



A triangle of velocity vectors

DERIVATION

1 By the **triangle law of vectors**, $\overrightarrow{P'A'} + \overrightarrow{A'B'} = \overrightarrow{P'B'}$. So the change in velocity is:

$$\overrightarrow{A'B'} = \overrightarrow{P'B'} - \overrightarrow{P'A'} = \vec{v}_2 - \vec{v}_1 = \Delta\vec{v}$$

- 2 When $\Delta t \rightarrow 0$, A' is very close to B' . Then $A'B'$ is almost an arc of a circle whose radius is $P'A' = |\vec{v}|$. So:

$$\Delta\theta = \frac{A'B'}{P'A'} = \frac{|\Delta\vec{v}|}{|\vec{v}|}$$

- 3 From Eq. (i), $\Delta\theta = \omega \Delta t$. Therefore:

$$\omega \Delta t = \frac{|\Delta\vec{v}|}{|\vec{v}|} \Rightarrow \omega |\vec{v}| = \frac{|\Delta\vec{v}|}{\Delta t}$$

- 4 Use $v = \omega r$:

$$\frac{|\Delta\vec{v}|}{\Delta t} = \omega |\vec{v}| = (\omega r) \omega = \omega^2 r$$

- 5 In the limit $\Delta t \rightarrow 0$, the ratio $\frac{|\Delta\vec{v}|}{\Delta t}$ is the size of the **centripetal acceleration** at P :

$$|\vec{a}| = \frac{|\Delta\vec{v}|}{\Delta t} = \omega^2 r = \left(\frac{v}{r}\right)^2 r = \frac{v^2}{r}$$

CENTRIPETAL ACCELERATION

$$a = \frac{v^2}{r} = r\omega^2 = v\omega$$

It points towards the centre of the circle.

NOTE

Radius of Curvature

A small piece of any curved path looks like a part of a circle. The radius of the circle that fits the curve best at a point is called the **radius of curvature** at that point. In $a = \frac{v^2}{r}$, the r stands for this **radius of curvature**.