

CHAPTER

13

Oscillations

Periodic Motion and SHM • Characteristics of SHM • Energy in SHM, Spring and Pendulum

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READING KEY

DEFINITION

NOTE

REMEMBER

DERIVATION

STRATEGY

Formulas sit on graph paper • coloured words are the key terms

An object is in **motion** when its **position** changes with **time**, as seen from a **reference point**. We can sort motions into types by the **path** they follow and by some other properties.

1.1 Periodic and Oscillatory Motion

THREE TYPES OF MOTION

i Non-repetitive

The motion **does not repeat**.

e.g. Rectilinear motion, projectile motion.

ii Periodic

The motion **repeats** after equal times.

e.g. Circular or orbital motion, a bouncing ball.

iii Oscillatory

The body moves **to and fro** about a point.

e.g. The motion of a pendulum.

DEFINITION

Periodic Motion

Periodic motion is a motion that **repeats itself** after **equal intervals of time**.

e.g. The planets revolve around the Sun. The Earth rotates about its polar axis.

DEFINITION

Oscillatory Motion

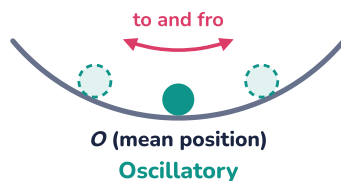
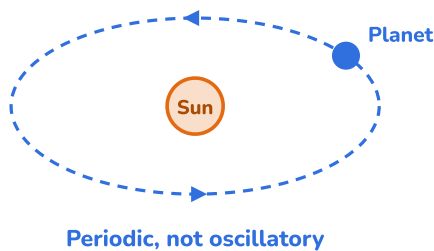
Oscillatory motion is the **to-and-fro** (back and forth) motion of a body about a **fixed point**, repeated at **regular intervals of time**.

This fixed point is called the **mean position**, the **equilibrium position** or the **centre of oscillation**.

e.g. A simple pendulum swings to and fro about its mean position.

• OSCILLATIONS AROUND US

- ◆ The strings of a **sitar** or a **guitar** vibrate.
- ◆ **Air molecules** vibrate and carry **sound** from place to place.
- ◆ **Atoms in a solid** vibrate about their equilibrium positions.
- ◆ An **AC voltage** keeps going positive and negative about its mean value.



A planet repeats its path, so its motion is periodic. A ball in a bowl moves to and fro about O, so its motion is oscillatory

REMEMBER

Periodic vs Oscillatory Motion

- ◆ **Every oscillatory motion is periodic.** But a periodic motion **need not be oscillatory**.
- ◆ **e.g.** A planet moving around the Sun is in periodic motion. It is **not oscillatory**, because it does not move to and fro.

NOTE

Equilibrium Position

An oscillating body has an **equilibrium position** on its path. At this position, **no net force** acts on the body. So a body placed there **at rest stays there for ever**.

DEFINITION

Restoring Force

The force that tries to bring a displaced body **back to its equilibrium position** is called the **restoring force**.

Give the body a **small displacement** from its mean position. The restoring force pulls it back. This happens again and again, and the body **oscillates**.

e.g. A ball rests at the bottom of a bowl in equilibrium. Push it a little to one side. It starts oscillating inside the bowl.

1.2 Period and Frequency

● PERIOD

DEFINITION

The **period** (time period) is the **smallest time interval** after which a periodic motion **repeats itself**. It is written as T . Its SI unit is the **second (s)**.

Periods can be **very short** or **very long**. So a **suitable unit** is chosen in each case:

- (i) Period of vibration of a **quartz crystal**, in microseconds ($1 \mu\text{s} = 10^{-6} \text{ s}$)

μs

(ii) The planet **Mercury** goes once round the Sun in

88 Earth days

(iii) **Halley's comet** takes 76 years for one trip round the Sun, so we see it

every 76 years

• FREQUENCY

DEFINITION

The **frequency** of an oscillating body is the **number of oscillations** it completes about its mean position in **unit time**. It is written as ν .

Its SI unit is the **hertz (Hz)**. The unit is named after **Heinrich Rudolph Hertz** (1857–1894), who discovered **radio waves**.

Frequency is also the **reciprocal of the time period**:

FREQUENCY

$$\nu = \frac{1}{T}$$

where, T = time period

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1}$$

DEFINITION

Vibrational Motion

An oscillatory motion of **very high frequency** is called **vibrational motion**.

1.3 Displacement

Displacement always means a **change in position** of a particle. Only the **reference point** is different in oscillations.

i In General

Displacement is the change in position of a particle measured from the **origin**.

ii In Oscillations

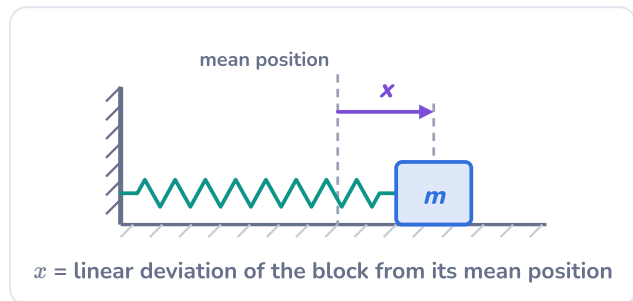
Displacement is the change in position measured from the **mean position** (equilibrium position). This point is also called the **reference position**.

1.4 Displacement Variable

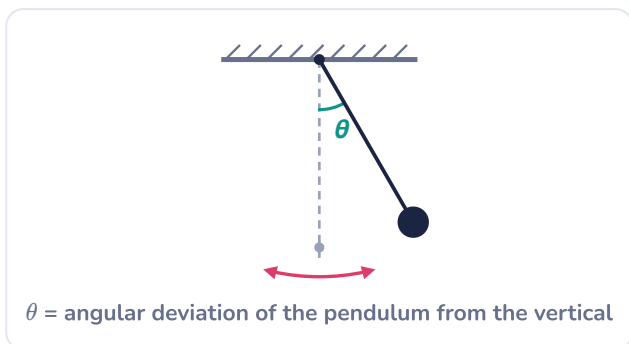
The quantity that measures the displacement during an oscillation is called the **displacement variable**. It changes with **time**. It takes both **positive (+)** and **negative (-)** values.

i Block on a Spring

A block is fixed to one end of a spring. The other end of the spring is fixed to a **rigid wall**. The displacement variable x is the **distance of the block from its mean position** at any time. It is a **linear deviation**.

**ii Simple Pendulum**

For a simple pendulum, the displacement variable is the **angle θ** that the string makes with the **vertical** at any time. It is an **angular deviation**.

**1.5 Periodic Function****DEFINITION**

We can write the displacement as a **mathematical function of time**. A function of time that **repeats its value** after a regular interval (its period) is called a **periodic function**.

The **simplest** periodic function is:

$$f(t) = A \cos \omega t \quad \dots (i)$$

Its period is $T = 2\pi/\omega$. The reason is simple. Adding 2π to the angle ωt does not change the cosine. So the function takes the **same value again** after a time T :

PERIODIC FUNCTION

$$f(t) = f(t + T), \quad T = \frac{2\pi}{\omega}$$

In the same way, the **sine function** $f(t) = A \sin \omega t$ is periodic. A **linear combination** of sine and cosine functions is also periodic, with the **same period T** :

$$f(t) = A \sin \omega t + B \cos \omega t \quad \dots (ii)$$

Now put:

Coefficient of $\sin \omega t$

$$A = D \cos \phi$$

Coefficient of $\cos \omega t$

$$B = D \sin \phi$$

Then Eq. (ii) takes a simpler form:

COMBINED PERIODIC FUNCTION

$$f(t) = D \sin(\omega t + \phi)$$

where, $D = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}\left(\frac{B}{A}\right)$

GOOD TO KNOW

Fourier's Result

The French mathematician **Jean Baptiste Joseph Fourier** (1768–1830) gave an important result. **Any periodic function** can be written as a **superposition of sine and cosine functions**. These functions have **different time periods** and **suitable coefficients**.

1.6 Simple Harmonic Motion

DEFINITION

Simple harmonic motion (SHM) is a special type of periodic motion. In it, a particle moves **to and fro** about a **mean position** under a **restoring force**. This force is **directly proportional to the displacement** and always acts **towards the mean position**.

• THE RESTORING FORCE IN SHM

i Direction

It always acts **towards the mean position**.

ii Magnitude

At every instant, its size is **directly proportional** to how far the particle is from the mean position.

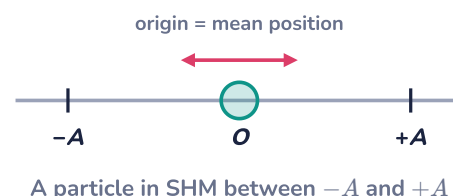
KEY

Because of such a restoring force, the displacement in SHM always changes with time as a sine or cosine, that is, as a sinusoidal function.

1.7 Equation of SHM

A particle moves back and forth along the x -axis about the **origin O**. It stays between the limits $+A$ and $-A$.

This motion is **SHM** if the displacement x of the particle from the origin changes with time as given below.



DISPLACEMENT IN SHM

$$x(t) = A \cos(\omega t + \phi)$$

where, A , ω and ϕ are **constants**

With $\phi = 0$: at $t = 0$, the particle is at

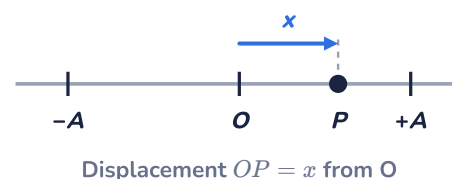
$$x = +A$$

At $t = T$, it comes back to the **same point**

$$x = +A$$

Let the particle be at point **P** at an instant t . Take O as the **mean position**.

Then $OP = x$ tells how far the particle is from the mean position. It is the **displacement** of the particle.



At that instant, a **restoring force** F acts on the particle:

FORCE LAW FOR SHM

$$F = -kx$$

where, k = **force constant**. Its SI unit is N m^{-1} (N/m).

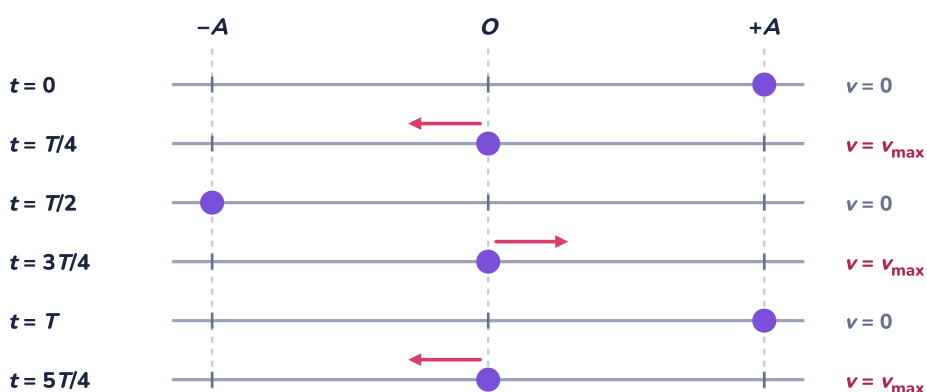
NOTE

Why the Minus Sign?

The minus sign shows that F always points **towards the mean position**. So F and x are always in **opposite directions**. The relation $F = -kx$ is called the **force law for SHM**.

1.8 Position of the Particle at Different Times

The figure shows where a particle in SHM is at **different times**. It starts at $+A$ at $t = 0$. Each step is a time interval of $\frac{T}{4}$.



Positions at $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T, \frac{5T}{4}$. The speed is zero at $\pm A$ and maximum at O

OBSERVATIONS

1 Repetition

The motion **repeats itself** after a time T .

2 Fixed Period

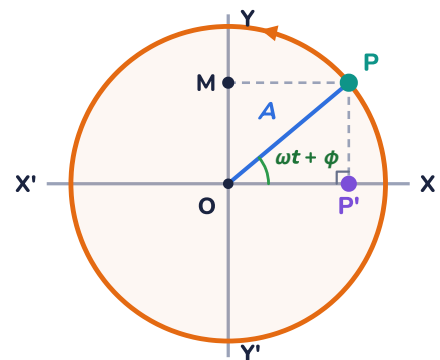
T **stays fixed**. It does not change if the **starting position** (at $t = 0$) is changed.

1.9 SHM and Uniform Circular Motion

This gives a geometrical picture of SHM.

A particle **P** starts from point **X**. It moves with **uniform speed** along a circle of **radius A** and centre **O**.

Draw a perpendicular from **P** to the diameter **XOX'**. Its foot is the point **P'**.



Circle of reference: **P'** is the foot of the perpendicular from **P** on **XOX'**

Circle of Reference

The circle on which **P** moves is called the **circle of reference**.

Reference Particle

The particle **P** is called the **reference particle**.

Projection

P' is called the **projection** of **P** on the diameter **XOX'**.

As **P** goes round the circle, its projection **P'** moves along the diameter as follows:

P MOVES ALONG THE CIRCLE	P' MOVES ALONG THE DIAMETER
$X \rightarrow Y$	$X \rightarrow O$
$Y \rightarrow X'$	$O \rightarrow X'$
$X' \rightarrow Y'$	$X' \rightarrow O$
$Y' \rightarrow X$	$O \rightarrow X$

RESULT

P goes **round the circle**. At the same time, **P'** moves **to and fro about O** along **XOX'**. So the motion of **P'** about **O** is **simple harmonic**.

DEFINITION

Geometrical Definition of SHM

The **projection of a uniform circular motion** on **any diameter** of its circle is a **simple harmonic motion**.

NOTE

The projection of the reference particle on **any other diameter** of the circle of reference also moves in **SHM**.

02 PART 2 Characteristics of SHM

A few **key quantities** describe every simple harmonic motion: **displacement**, **amplitude**, **phase**, **angular frequency**, **velocity** and **acceleration**.

2.1 Displacement in SHM

DEFINITION

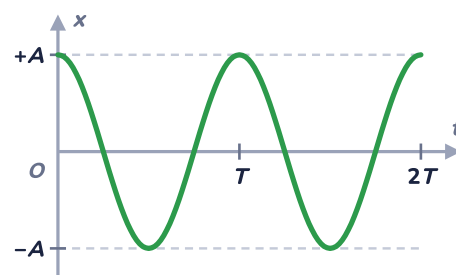
The **displacement** of a particle in SHM at any instant is its **distance from the mean position** at that instant.

The displacement changes **continuously with time**. So we can draw it as a graph.

The graph swings between $+A$ and $-A$ about O.

DISPLACEMENT

$$x(t) = A \cos(\omega t + \phi)$$



Displacement as a continuous function of time

DEFINITION

Sinusoidal Function

A function that contains a **sine** or a **cosine** term is called a **sinusoidal function**.

2.2 Amplitude

DEFINITION

The **amplitude** of a particle in SHM is the **magnitude of its maximum displacement** from the mean position. It is measured on **either side** of the mean position.

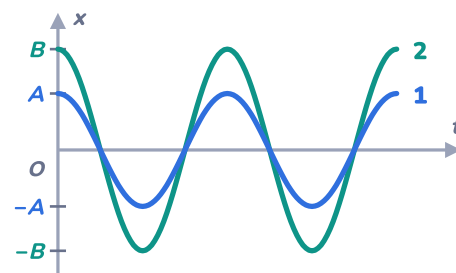
Two SHMs can have the **same** angular frequency ω and the **same** phase constant ϕ . But their **amplitudes** A and B can be **different**.

Same

ω, ϕ

Different

amplitudes A, B



Same ω and ϕ , different amplitudes A and B

sin or cos goes from +1 to -1 $\Rightarrow x$ goes from $+A$ to $-A$

2.3 Phase

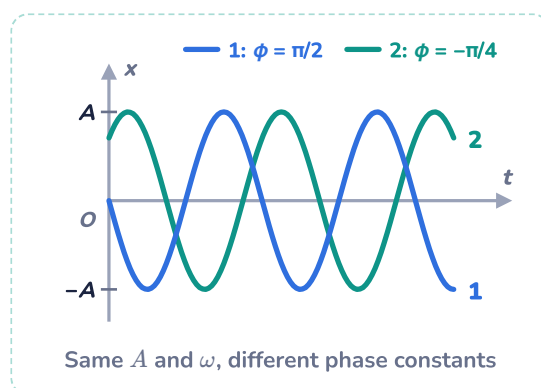
DEFINITION

The **phase** of an SHM is the quantity $(\omega t + \phi)$ inside its sinusoidal function. For a fixed amplitude A , the phase decides the **state of motion**, that is, the **position** and the **velocity** of the particle at time t .

At $t = 0 \Rightarrow \text{phase} = \phi \Rightarrow \phi$ is the phase constant (phase angle)

Two SHMs can have the **same** amplitude A and the **same** angular frequency ω . But their **phase constants** ϕ can be **different**.

In the graph, **curve 1** has $\phi = \frac{\pi}{2}$. **Curve 2** has $\phi = -\frac{\pi}{4}$. Both curves have amplitude A .



2.4 Angular Frequency

DEFINITION

The **angular frequency** of a periodic motion is 2π **times its frequency**. It is written as ω . Its SI unit is the **radian per second** (rad s^{-1}).

$$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T} = 2\pi\nu, \quad \text{where } \nu = \text{frequency of the particle}$$

DERIVATION

Why is $\omega = 2\pi/T$?

The motion repeats after one period T . So the displacement must come back to its **first value** after T , that is, $x(t) = x(t + T)$ for every t . Take $x(t) = A \cos \omega t$:

1

$$A \cos \omega t = A \cos \omega(t + T)$$

2

The cosine first repeats when its angle grows by 2π . So:

$$\omega(t + T) = \omega t + 2\pi$$

3

$$\omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T}$$

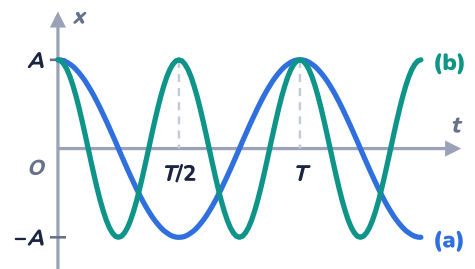
ANGULAR FREQUENCY

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

ω is 2π times the frequency of oscillation, $\nu = \frac{1}{T}$

Two SHMs can have the **same** amplitude A and the **same** phase constant ϕ . But their **angular frequencies** ω can be **different**.

In the graph, **curve (b)** has **half the period** of **curve (a)**. So it has **twice the frequency** of curve (a).



Same A and ϕ : curve (b) has half the period of curve (a)

2.5 Velocity in SHM

DEFINITION

The **velocity** of a particle in SHM at any instant is the **rate of change of its displacement with time** at that instant.

VELOCITY IN TERMS OF DISPLACEMENT

$$v = \omega \sqrt{A^2 - x^2}$$

This gives the magnitude of the velocity (the speed) at displacement x .

At the Mean Position ($x = 0$)

$$v = \omega A$$

The velocity is **maximum**.

At an Extreme Position ($x = \pm A$)

$$v = 0$$

The velocity is **minimum** (zero).

We can also find the velocity by **differentiating** $x(t)$:

$$v(t) = \frac{d}{dt}[x(t)] = \frac{d}{dt}[A \cos(\omega t + \phi)]$$

$$v(t) = -\omega A \sin(\omega t + \phi)$$

2.6 Acceleration in SHM

DEFINITION

The **acceleration** of a particle in SHM at any instant is the **rate of change of its velocity with time** at that instant.

ACCELERATION IN TERMS OF DISPLACEMENT

$$a = -\omega^2 x$$

At the Mean Position ($x = 0$)

$$a = 0$$

The acceleration is **minimum** (zero).

At an Extreme Position ($x = \pm A$)

$$a = \mp \omega^2 A$$

The acceleration is **maximum** in size.

REMEMBER

Velocity and Acceleration Behave Oppositely

- ◆ At the **mean position**: velocity is **maximum**, acceleration is **zero**.
- ◆ At an **extreme position**: velocity is **zero**, acceleration is **maximum**.

We can also find the acceleration by **differentiating** $v(t)$:

$$a(t) = \frac{d}{dt}[v(t)] = \frac{d}{dt}[-\omega A \sin(\omega t + \phi)]$$

ACCELERATION AS A FUNCTION OF TIME

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

2.7 Graphs of Displacement, Velocity and Acceleration

Take $\phi = 0$. Then the three quantities of a body in SHM are:

DISPLACEMENT

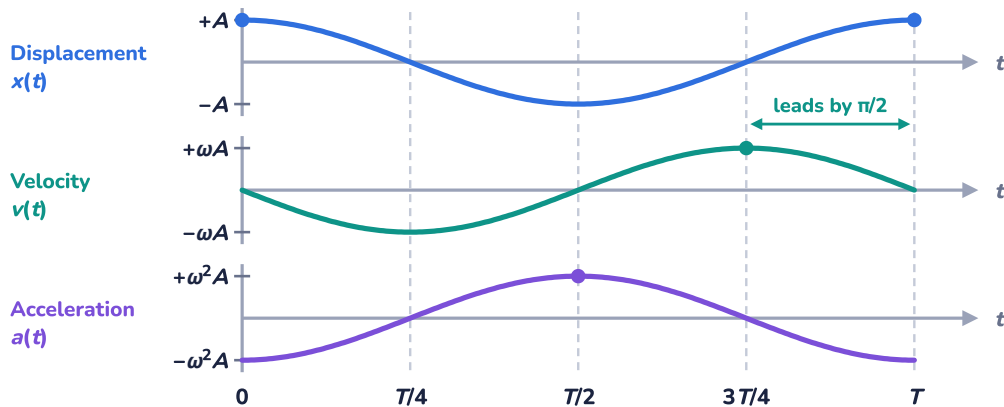
$$x(t) = A \cos \omega t$$

VELOCITY

$$v(t) = -\omega A \sin \omega t$$

ACCELERATION

$$a(t) = -\omega^2 A \cos \omega t$$



The three graphs share one time axis. v reaches each value $\frac{T}{4}$ earlier than x (it leads by $\frac{\pi}{2}$), and a is always opposite to x

QUANTITY	RANGE OF VALUES	PHASE DIFFERENCE FROM $x(t)$
Displacement $x(t)$	$-A$ to $+A$	Zero (reference curve)
Velocity $v(t)$	$-\omega A$ to $+\omega A$	$\frac{\pi}{2}$ (v leads x)
Acceleration $a(t)$	$-\omega^2 A$ to $+\omega^2 A$	π (a is opposite to x)

REMEMBER

- ◆ All three graphs are **sinusoidal** in nature.
- ◆ $v(t) = \omega A \cos\left(\omega t + \frac{\pi}{2}\right)$. So the velocity **leads** the displacement by $\frac{\pi}{2}$.
- ◆ $a(t) = \omega^2 A \cos(\omega t + \pi)$. So the acceleration is always **opposite** to the displacement.

2.8 Force Law for SHM

Move a body a little away from its **equilibrium (mean) position**. A **restoring force** then acts on it, **opposite to the displacement**. It tries to bring the body back to equilibrium. For a **small displacement**, this force is **proportional to the displacement**. If the body is left free, it moves back towards the mean position.

On the way back, the body gains **kinetic energy**. So it does not stop at the mean position. It **overshoots** and goes to the **other side**. There, a restoring force again **slows it down** and brings it back. This keeps repeating.

Displaced → pulled back → gains KE and overshoots → other side → repeats

In this way, the restoring force makes the body **oscillate** about the mean position with a **definite period**.

DERIVATION

By **Newton's second law**, $F = ma$. Put $a(t) = -\omega^2 x(t)$ and write the constant $m\omega^2$ as k :

1

$$F(t) = -m\omega^2 x(t) = -kx(t) \quad \dots (i)$$

$$F(t) = -kx(t)$$

$$\text{where, } k = m\omega^2 \text{ or } \omega = \sqrt{\frac{k}{m}}$$

DEFINITION

Alternative Definition of SHM

Eq. (i) gives **another definition** of SHM. **Simple harmonic motion** is the motion of a particle under a force that is **proportional to its displacement** and is always directed **towards the mean position**.

03 PART 3 Energy in SHM, Spring and Pendulum

A particle in SHM has both **kinetic energy** and **potential energy**. This part finds both energies. It then studies two real systems that move in SHM: a **spring-block system** and a **simple pendulum**.

3.1 Kinetic Energy in SHM

To move a body away from its equilibrium position, we must **do work** on it. This work gives the body **potential energy**. When we release it, the body moves back with some **velocity**. So it gains **kinetic energy**.

Work done to displace → gains PE → released → moves back, gains KE

NOTE

In SHM, the **kinetic energy** and the **potential energy** both keep changing between **zero** and their **maximum values**.

DERIVATION

Kinetic Energy

At any instant, the displacement of the particle is $x = A \cos(\omega t + \phi)$. So its velocity is:

1

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

2

Kinetic energy of the particle at displacement x :

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

3 Since $k = m\omega^2$:

$$K = \frac{1}{2}kA^2 \sin^2(\omega t + \phi)$$

4 Now use $\sin^2 = 1 - \cos^2$ and $x = A \cos(\omega t + \phi)$:

$$A^2 \sin^2(\omega t + \phi) = A^2[1 - \cos^2(\omega t + \phi)]$$

$$= A^2 - A^2 \cos^2(\omega t + \phi) = A^2 - x^2$$

5 So, in terms of displacement:

$$K = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

KINETIC ENERGY IN SHM

$$K = \frac{1}{2}k(A^2 - x^2)$$

K also changes **periodically with time**. It becomes **zero** when the displacement is

maximum

K has its **largest value** at the

mean position

Period of the kinetic energy

$T/2$

3.2 Potential Energy in SHM

DERIVATION

Potential Energy

Let the displacement of the particle from its equilibrium position be x . The restoring force on it is $F = -kx$.

1 Move the particle further by a **tiny distance** dx . The work done by the restoring force is:

$$dW = F dx = -kx dx$$

2 Total work done by this force as the particle moves from the mean position ($x = 0$) to x :

$$W = \int dW = - \int_0^x kx dx = -k \left[\frac{x^2}{2} \right]_0^x = -\frac{1}{2}kx^2$$

3 The **work done against** the restoring force is $-W$. It is **stored as potential energy**:

$$U = -W = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

POTENTIAL ENERGY IN SHM

$$U = \frac{1}{2}kx^2$$

Put $x = A \cos(\omega t + \phi)$ to write U in terms of **time**:

$$U = \frac{1}{2}m\omega^2A^2 \cos^2(\omega t + \phi) = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

U is also **periodic**, with period

$$T/2$$

U is **zero** at the

mean position

U is **maximum** at an

extreme position

REMEMBER

KE and PE Behave Oppositely

- ◆ At the **mean position**: KE is **maximum**, PE is **zero**.
- ◆ At an **extreme position**: KE is **zero**, PE is **maximum**.
- ◆ Both energies have the **same period**, $T/2$.

3.3 Total Energy in SHM

At any displacement x , the **total energy** of the oscillating particle is:

$$E = K + U = \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}kx^2$$

TOTAL ENERGY IN SHM

$$E = \frac{1}{2}kA^2$$

$$E = \frac{1}{2}m\omega^2A^2 = 2\pi^2m\nu^2A^2 \quad [\because \omega = 2\pi\nu]$$

Does Not Depend On

The total mechanical energy of an SHM does **not** **depend on time or on displacement**.

Depends On

It depends on the **maximum displacement**, that is, the **amplitude**.

PRINCIPLE

Conservation of Energy

So, when there is **no friction**, the total energy of an SHM is **conserved**.

So the total energy E varies **directly** as:

its **mass**

$$E \propto m$$

the **square of the frequency**

$$E \propto \nu^2$$

the **square of the amplitude**

$$E \propto A^2$$

3.4 Graphs of Energy in SHM

At the Mean Position ($x = 0$)

(i) Kinetic energy, $K = \frac{1}{2}k(A^2 - 0^2) = \frac{1}{2}kA^2$

(ii) Potential energy, $U = \frac{1}{2}k(0^2) = 0$

All the mechanical energy is **kinetic**.

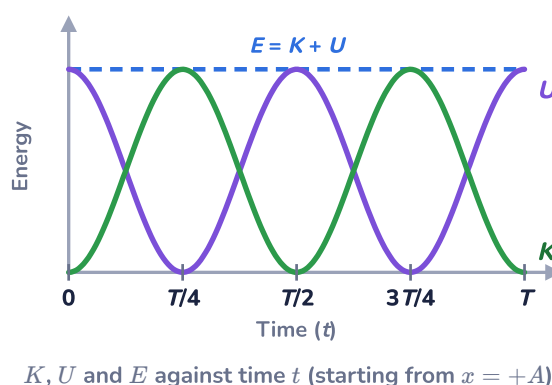
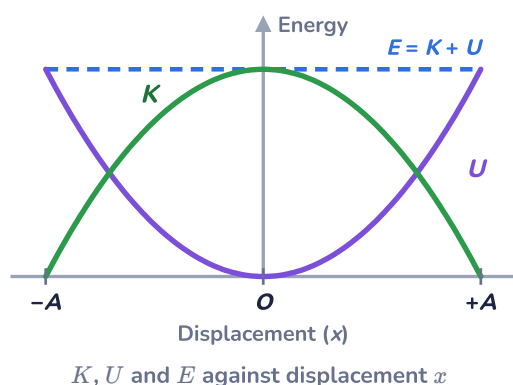
At the Extreme Positions ($x = \pm A$)

(i) Kinetic energy, $K = \frac{1}{2}k(A^2 - A^2) = 0$

(ii) Potential energy, $U = \frac{1}{2}kA^2$

All the energy is **potential**. The KE is zero.

Using these results, we can draw K , U and E against the **displacement** x and against the **time** t :



PE or KE completes **two vibrations** while the SHM completes **one** (each repeats after $T/2$)

2 : 1

So the frequency of PE or KE is

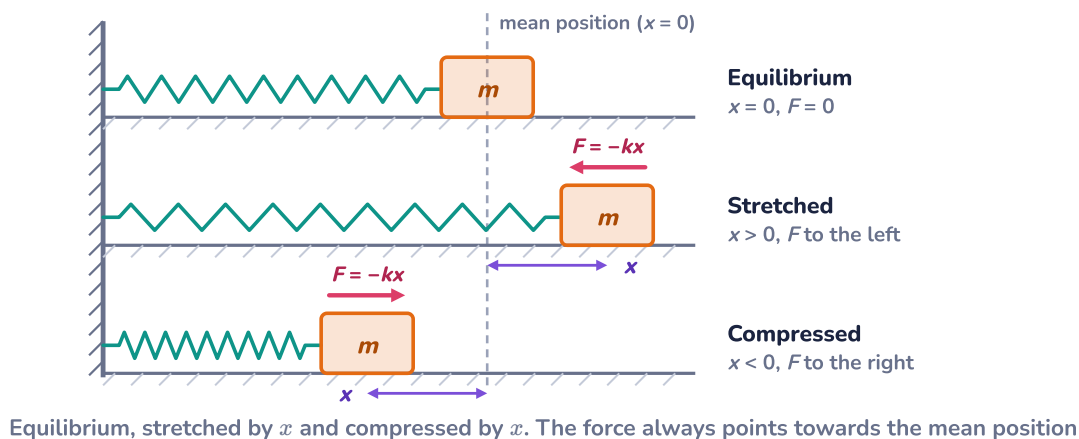
double that of SHM

KEY The total energy stays constant at every time t and every displacement x .

3.5 Oscillations of a Spring

A **pure SHM** is possible only under some **conditions**. Two examples of pure SHM under such conditions are a spring–block system and a simple pendulum.

Take a **massless spring** on a **frictionless horizontal table**. One end is fixed to a **rigid support**. A **block of mass m** is fixed to the other end. Pull the block to the **right** by a **small distance x** and release it. The restoring force then makes it **move back and forth** about its equilibrium position.



$$F = -kx$$

NOTE

Force Constant k

k is the **force constant** of the spring. It is the **restoring force per unit deformation** of the spring. The minus sign shows that the force acts **opposite to x** .

DERIVATION

Time Period of a Spring–Block System

Let $\frac{d^2x}{dt^2}$ be the acceleration of the block. By Newton's second law, $F = ma = m\frac{d^2x}{dt^2}$. Equate this with the restoring force:

1

$$m\frac{d^2x}{dt^2} = -kx$$

2

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2x$$

3

So the acceleration is **proportional to x** and acts **opposite to it**. Hence the block moves in **simple harmonic motion**.

4 Its time period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$$

TIME PERIOD

$$T = 2\pi\sqrt{\frac{m}{k}}$$

FREQUENCY

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

A very **stiff spring** (large k) with a **light body** (small m)

T small, ν large

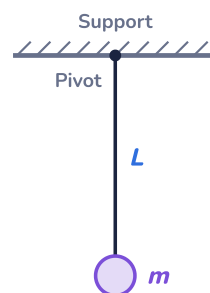
3.6 Simple Pendulum

DEFINITION

Ideal Simple Pendulum

An **ideal simple pendulum** is a **point mass** hung from a fixed support by an **inextensible, weightless string**. Such a pendulum **cannot be made in practice**.

In practice, we hang a **small metal bob** from a support with a **long, fine cotton thread**. This works as a simple pendulum.



A simple pendulum

i Ideal

Point mass + weightless string.

Not possible in practice.

ii Practical

Small metal bob + fine cotton thread.

The real-life version.

3.7 Time Period of a Simple Pendulum

A small bob of **mass** m hangs from an inextensible, massless string of **length** L . At equilibrium, the bob is **vertically below** the point of suspension.

Displace the bob **slightly** to one side and release it. It oscillates about the mean position O. At some instant, let the bob be at A. Its displacement along the arc is $OA = x$. The string makes an **angle** θ with the vertical.

i Weight

The **weight** mg of the bob acts **vertically downwards**.

ii Tension

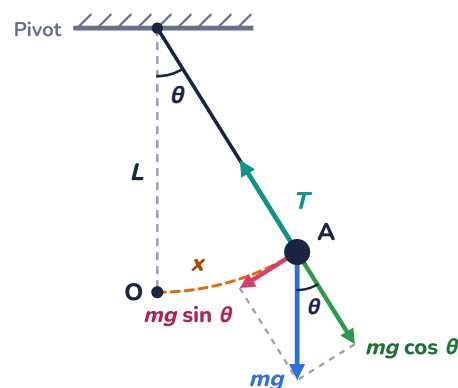
The **tension** T acts **along the string**, towards the support.

TWO RECTANGULAR COMPONENTS OF MG $mg \cos \theta$ — along the string

It is **balanced by the tension** T in the string.

 $mg \sin \theta$ — along the tangent

It gives a **restoring torque**. It tries to bring the bob **back to the mean position**.



Forces on the bob: weight mg and tension T . mg splits into $mg \cos \theta$ and $mg \sin \theta$

DERIVATION

Time Period of a Simple Pendulum

Let L = length of the simple pendulum, I = moment of inertia of the pendulum about the pivot, and α = angular acceleration about the pivot.

- 1 Restoring torque of $mg \sin \theta$ about the pivot. The minus sign shows that the torque acts to **reduce** θ :

$$\tau = -(mg \sin \theta)L = -mgL \sin \theta \quad \dots (i)$$

- 2 For rotation, the torque is also:

$$\tau = I\alpha \quad \dots (ii)$$

- 3 From Eqs. (i) and (ii):

$$-mgL \sin \theta = I\alpha \Rightarrow \alpha = \frac{-mgL \sin \theta}{I}$$

- 4 With θ in radian, $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$ For a **small** θ , $\sin \theta \approx \theta$. So:

$$\alpha = \frac{-mgL \theta}{I}$$

5 This is the **angular form** of $a = -\omega^2 x$. Comparing the two:

$$\omega = \sqrt{\frac{mgL}{I}}$$

6 Since $\omega = \frac{2\pi}{T}$:

$$\frac{2\pi}{T} = \sqrt{\frac{mgL}{I}} \Rightarrow T = 2\pi \sqrt{\frac{I}{mgL}} \quad \dots \text{(iii)}$$

7 The whole mass of a simple pendulum is in the bob, taken as a point at distance L from the pivot. So $I = mL^2$. Put this in Eq. (iii):

$$T = 2\pi \sqrt{\frac{mL^2}{mgL}} = 2\pi \sqrt{\frac{L}{g}}$$

NOTE

Why Is It SHM?

The angular acceleration is **proportional to the angular displacement** θ , but it has the **opposite sign**. So when the bob moves to the right, the pull towards the left grows. The bob stops and comes back. The same thing happens on the left side. So the bob swings to and fro. Hence a simple pendulum swinging through **small angles** moves in **approximately SHM**.

TIME PERIOD OF A SIMPLE PENDULUM

$$T = 2\pi \sqrt{\frac{L}{g}}$$

T **depends on** the length L and the acceleration due to gravity g

depends

T does **not depend on the mass** m of the bob

independent

$T = 2\pi \sqrt{L/g}$ holds only when L is **small compared with the Earth's radius**

small L only

IMPORTANT

If the Length Is Very Large

$$T = 2\pi \sqrt{\frac{R}{(1 + \frac{R}{L})g}}, \quad \text{where } R = \text{radius of the Earth}$$

$$\text{As } L \rightarrow \infty : T = 2\pi \sqrt{\frac{R}{g}} \quad \text{When } L = R : T = 2\pi \sqrt{\frac{R}{2g}}$$

TERM	KEY POINTS
Periodic motion	Repeats itself after regular intervals of time.
Oscillatory motion	To-and-fro motion, again and again, about a fixed point (the mean position). Also called vibratory motion .
SHM	Periodic motion in which the restoring force is directly proportional to the displacement from the mean position: $F \propto x$ or $F = -kx$. Here k = force constant (spring constant).
Displacement	$x = A \cos(\omega t + \phi)$, or in sine form $y = A \sin(\omega t + \phi)$. A = amplitude, ω = angular frequency $\left(\omega = \frac{2\pi}{T} = 2\pi\nu\right)$, ϕ = initial phase.
Time period T	Time taken to complete one oscillation about the mean position.
Frequency ν	Oscillations completed per second. $\nu = \frac{1}{T}$, in hertz. $1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1}$.
Angular frequency ω	Frequency multiplied by 2π : $\omega = 2\pi\nu = \frac{2\pi}{T}$. SI unit: rad s^{-1} .
Phase	Gives the state of the particle at any instant: its position and its direction of motion.
Velocity	$v = \omega\sqrt{A^2 - x^2}$ or $v(t) = -\omega A \sin(\omega t + \phi)$. At the mean position ($x = 0$): $v = A\omega = v_{\max}$ (velocity amplitude). At $x = \pm A$: $v = 0$.
Acceleration	$a = -\omega^2 x$ or $a(t) = -\omega^2 A \cos(\omega t + \phi)$. At $x = 0$: $a = 0$. At $x = \pm A$: $a_{\max} = \mp A\omega^2$ (acceleration amplitude = $A\omega^2$).
Time period in SHM	$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left \frac{x}{a}\right } = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$
Energy in SHM	$U = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}kx^2$ $K = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$ $E = U + K = \frac{1}{2}m\omega^2 A^2 = 2\pi^2 m\nu^2 A^2$
Simple pendulum	In practice: a heavy but small metal bob hung by a light, inextensible, flexible string. $T = 2\pi\sqrt{\frac{L}{g}}$ and $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.